GEORGIA INSTITUTE OF TECHNOLOGY

SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 6250 Spring 2017

Problem Set #2

Assigned: 23-Jan-17
Due Date: 1-Feb-17

This assignment is due in class on Wednesday, February 1.

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

PROBLEM 2.1:
Using you class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

PROBLEM 2.2:
Prove the "reverse triangle inequality": show that in a normed linear space

$$\|\boldsymbol{x}\| - \|\boldsymbol{y}\| \ \leq \ \|\boldsymbol{x} - \boldsymbol{y}\|.$$

What can we say about $|\ \|\boldsymbol{x}\| - \|\boldsymbol{y}\|\ |$ in relation to $\|\boldsymbol{x} - \boldsymbol{y}\|$?

PROBLEM 2.3:
Let $\|\cdot\|_p$ be the ℓ_p norm for vectors in $\mathbb{R}^N$ as defined in the notes.

1. Prove that for any integer $p \geq 1$

$$\|\boldsymbol{x}\|_p \leq \|\boldsymbol{x}\|_1 \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^N$$

2. Prove that for *any* $1 \leq q \leq p \leq \infty$

$$\|\boldsymbol{x}\|_p \leq \|\boldsymbol{x}\|_q \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^N.$$

(Hint: If a_1 and a_2 are non-negative real numbers, then $(a_1 + a_2)^\alpha \geq a_1^\alpha + a_2^\alpha$ for $\alpha \geq 1$.)

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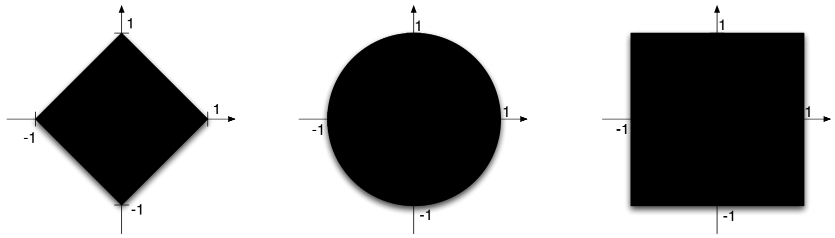
PROBLEM 2.4:
One way to visualize a norm in $\mathbb{R}^2$ is by its *unit ball*, the set of all vectors such that $\|\boldsymbol{x}\| \leq 1$. For example, we have seen that the unit balls for the ℓ_1, ℓ_2, and ℓ_∞ norms look like:



$B_1 = \{\boldsymbol{x} : \|\boldsymbol{x}\|_1 \leq 1\}$ $B_2 = \{\boldsymbol{x} : \|\boldsymbol{x}\|_2 \leq 1\}$ $B_\infty = \{\boldsymbol{x} : \|\boldsymbol{x}\|_\infty \leq 1\}$

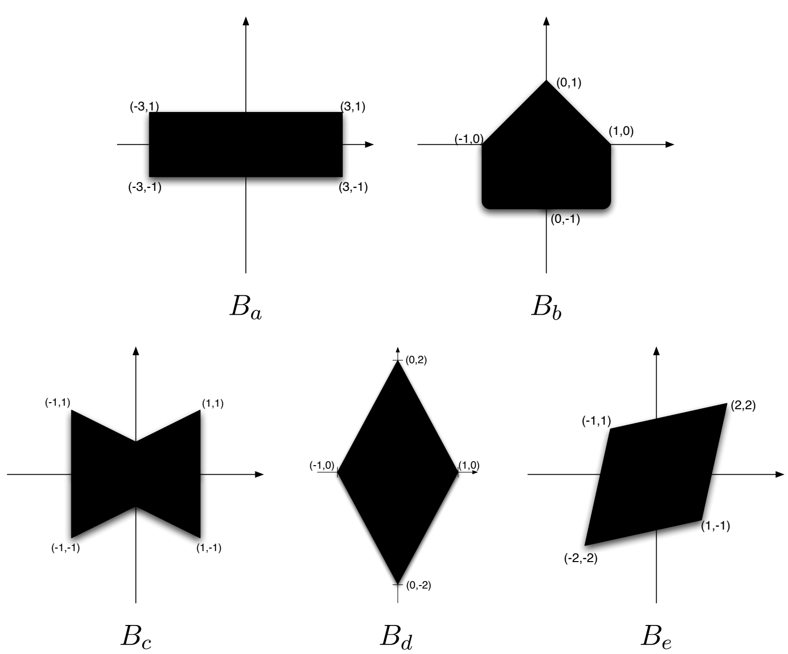
Given an appropriate subset of the plane, $B \subset \mathbb{R}^2$, it might be possible to define a corresponding norm using

$$\|\boldsymbol{x}\|_B = \text{minimum value } r \geq 0 \text{ such that } \boldsymbol{x} \in rB, \tag{1}$$

where rB is just a scaling of the set B:

$$\boldsymbol{x} \in B \quad \Rightarrow \quad r \cdot \boldsymbol{x} \in rB.$$

1. Let $\boldsymbol{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. For $p = 1, 2, \infty$, find $r = \|\boldsymbol{x}\|_p$, and sketch $\boldsymbol{x}$ and rB_p (use different axes for each of the three values of p).
2. Consider the 5 shapes below.



For each, explain why the corresponding $\|\cdot\|_B$ defined in (1) is or is not a valid norm.

3. For the B that are valid norms, give a concrete method for computing $\|\boldsymbol{x}\|_B$. (For example: for B_1, we would write $\|\boldsymbol{x}\|_1 = |x_1| + |x_2|$.)
This will require a little bit of thought.

PROBLEM 2.5:
Below, $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^N$.

1. Prove that $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leq \|\boldsymbol{x}\|_\infty \cdot \|\boldsymbol{y}\|_1$.

2. Prove that $\|\boldsymbol{x}\|_1 \leq \sqrt{N} \cdot \|\boldsymbol{x}\|_2$. (Hint: Cauchy-Schwarz)

3. Let B_2 be the unit ball for the ℓ_2 norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on $\boldsymbol{y}$:

$$\max_{\boldsymbol{x} \in B_2} \langle \boldsymbol{x}, \boldsymbol{y} \rangle \quad = \quad ????$$

   Describe the vector x which achieves the maximum. (Hint: Cauchy-Schwarz)

4. Let B_∞ be the unit ball for the ℓ_∞ norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on $\boldsymbol{y}$:

$$\max_{\boldsymbol{x} \in B_\infty} \langle \boldsymbol{x}, \boldsymbol{y} \rangle \quad = \quad ????$$

   Describe the vector $\boldsymbol{x}$ which achieves the maximum. (Hint: Part (a))

5. Let B_1 be the unit ball for the ℓ_1 norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on $\boldsymbol{y}$:

$$\max_{\boldsymbol{x} \in B_1} \langle \boldsymbol{x}, \boldsymbol{y} \rangle \quad = \quad ????$$

   Describe the vector $\boldsymbol{x}$ which achieves the maximum. (Hint: Part (a))

(These last two might require some thought. If you solve them for $N = 2$, it should be easy to generalize.)

PROBLEM 2.6:
Let $\boldsymbol{A}$ be the 2×2 matrix

$$\boldsymbol{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 4 & 4 \\ -1/2 & 1/2 \end{bmatrix}.$$

For $\boldsymbol{x} \in \mathbb{R}^2$, define $\|\boldsymbol{x}\|_A = \|\boldsymbol{A}\boldsymbol{x}\|_2$.

1. Show that $\|\cdot\|_A$ is indeed a valid norm.

2. Sketch the unit ball $B_A = \{\boldsymbol{x} : \|\boldsymbol{x}\|_A \leq 1\}$ corresponding to $\|\cdot\|_A$.

PROBLEM 2.7:

As we discussed in class, there is a natural way to map mth order polynomials to vector in $\mathbb{R}^{m+1}$:

$$x(t) = a_m t^m + a_{m-1}t^{m-1} + \cdots + a_1 t + a_0 \quad \longrightarrow \quad \begin{bmatrix} a_m \\ a_{m-1} \\ \vdots \\ a_0 \end{bmatrix},$$

so $3t^2 - t + 4$ is represented by $\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$, etc.

Of course, the derivative of an mth order polynomial is a polynomial of order $(m-1)$, which can then be represented as a vector in $\mathbb{R}^m$. For example, if

$$\boldsymbol{a}_x = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \quad \text{then} \quad \boldsymbol{D}[\boldsymbol{a}_x] = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

since the derivative of $3t^2 - t + 4$ is $6t - 1$. We have used $\boldsymbol{D}[\cdot]$ above to represent the derivative operator.

Taking the derivative is a linear operation, so this means that $\boldsymbol{D}[\cdot]$ can be represented by a $m \times (m+1)$ matrix.

1. Write a MATLAB function called `derivative_matrix` that takes a parameter m and returns the $m \times (m+1)$ matrix corresponding to the derivative operator.

2. Test out your code to solve the following exercise: Let

$$x(t) = 5.8t^5 - 9.2t^4 + 3.1t^3 - 6.2t^2 + 7.2t + 1,$$

and let $x'(t)$ be the derivative of $x(t)$.
Compute $x'(t)$ at the four points $t = 0.01, 0.45, 0.63, 0.81$.
(You will find the MATLAB function `polyval` useful here.)