GEORGIA INSTITUTE OF TECHNOLOGY

SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 6250 Spring 2017

Problem Set #5

Assigned: 17-Feb-17

Due Date: 8-Mar-17

This assignment is due at the beginning of class on Wednesday, March 8.

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

PROBLEM 5.1:

Using you class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

PROBLEM 5.2:

Write two MATLAB functions, called `mydct.m` and `myidct.m`, that implement the discrete cosine transform (DCT) and its inverse for a vector of length N. You code should be short (4 lines or less per function, no loops), efficient (it should make use of the `fft` command), and match the output of MATLAB's `dct` and `idct` commands. You can verify the latter with

```
x = randn(1000000,1);
d1 = mydct(x);
d2 = dct(x);
norm(d1-d2)

y = randn(1000000,1);
w1 = myidct(y);
w2 = idct(y);
norm(w1-w2)
```

The norms of the differences should be small in both cases. (The one thing you really have to handle here is the fact that the DCT uses cosines that are shifted by half a sample.)

Last updated 13:24, February 17, 2017

PROBLEM 5.3:
Let $\mathcal{E}$ be the space of signals on $[-1,1]$ that are even:

$$x(t) \in \mathcal{E} \quad \Leftrightarrow \quad x(t) = x(-t), \quad t \in [-1,1],$$

and let $\mathcal{O}$ be the space of signal son $[-1,1]$ that are odd:

$$x(t) \in \mathcal{O} \quad \Leftrightarrow \quad x(t) = -x(-t), \quad t \in [-1,1].$$

1. Given an arbitrary $x(t) \in L_2([-1,1])$ what is the closest even function to $\boldsymbol{x}$? That is, solve

$$\min_{\boldsymbol{y} \in \mathcal{E}} \ \|\boldsymbol{x} - \boldsymbol{y}\|_2^2.$$

A good way to do this is to use the *orthogonality principle* — we know that for the optimal $\hat{\boldsymbol{y}} \in \mathcal{E}$

$$\langle \boldsymbol{x} - \hat{\boldsymbol{y}}, \boldsymbol{z} \rangle = 0 \quad \text{for all} \quad \boldsymbol{z} \in \mathcal{E}.$$

You might consider filling in the blanks in the following line of reasoning:

$$\begin{aligned}
\langle \boldsymbol{x} - \hat{\boldsymbol{y}}, \boldsymbol{z} \rangle &= \int_{-1}^{1} [x(t) - y(t)] z(t) \ dt \\
&= \int_{0}^{1} [x(t) - y(t)] z(t) + \cdots \ dt \\
&= \cdots \\
&= 0 \quad \text{for all } \boldsymbol{z} \in \mathcal{E} \text{ when } \hat{y}(t) = \cdots .
\end{aligned}$$

2. Given an arbitrary $x(t) \in L_2([-1,1])$, solve

$$\min_{\boldsymbol{y} \in \mathcal{O}} \ \|\boldsymbol{x} - \boldsymbol{y}\|_2^2.$$

3. Let $\{\phi_k(t), \ k \geq 0\}$ be an orthobasis for $L_2([0,1])$. How can we use this orthobasis on $[0,1]$ to construct an orthobasis $\{\phi_k^e(t), \ k \geq 0\}$ for $\mathcal{E}$? What about an orthobasis $\{\phi_k^o(t), \ k \geq 0\}$ for $\mathcal{O}$? Is $\{\phi_k^e(t), \ k \geq 0\} \cup \{\phi_k^o(t), \ k \geq 0\}$ an orthobasis for all of $L_2([-1,1])$? Why or why not?

PROBLEM 5.4:

Implement the Haar wavelet transform and its inverse in MATLAB. Do this by writing two MATLAB functions, `haar.m` and `ihaar.m` that are called as `w=haar(x,L)` and `x = ihaar(w,L)`. Here, `x` is the original signal, and `L` is the number of levels in the transform.

You may assume that the length of `x` is dyadic; that is, the length of the input is 2^J for some positive integer J. In this situation, the Haar transform (no matter what L is) will have exactly 2^J terms, so the length of `x` and `w` should be the same.

If we interpret the input `x` as being scaling coefficients at scale J, then the vector `w` should consist of the scaling coefficients at scale $J-L$ stacked on top of the wavelet coefficients for scale $J-L$ stacked on top of the wavelet coefficients for scale $J-L+1$, etc.

Try your transform out on the data in `blocks.mat` and `bumps.mat`. For these two inputs, take a Haar wavelet transform with $L=3$ levels, and plot the scaling coefficients at scale $J-3$, and the wavelet coefficients at scales $J-3$ down to $J-1$. (For both of these signals, $J=10$.)

Also, verify that your transform is energy preserving.

Turn in printouts of your code along with the plots mentioned above.

PROBLEM 5.5:

1. Let $b[n]$ be an FIR filter of length L. Show how
$$\sum_{n=1}^{L} n^q b[n] = 0 \quad \text{for all} \quad q = 0, \ldots, p \tag{1}$$
implies
$$b[n] \star x[n] = \sum_{k=-\infty}^{\infty} b[k]x[n-k] = 0,$$
when
$$x[n] = a_p n^p + a_{p-1}n^{p-1} + \cdots + a_1 n + a_0$$
for arbitrary $a_p, \ldots, a_0 \in \mathbb{R}$. That is, show that the discrete sequence $b[n]$ having p vanishing moments really does mean that it annihilates (discrete) polynomials of order p.

2. Now suppose we have a wavelet $\psi_0(t)$, which can be written as a superposition of scaling functions at scale $j=1$ using
$$\psi_0(t) = \sum_n b[n]\phi_{1,n}(t).$$
Show that if the discrete *sequence* $b[n]$ has p vanishing moments (as in (1)), then the continuous time wavelet $\psi_0(t)$ must also have p vanishing moments, meaning
$$\int_{-\infty}^{\infty} t^q \psi_0(t)\ dt = 0 \quad \text{for all} \quad q = 0, \ldots, p.$$
Note that the $\phi_{1,n}(t)$ will not in general have vanishing moments — just make the following constant substitutions when you see the integrals below:
$$C_0 = \int \phi_{1,0}(t)\ dt, \quad C_1 = \int t\phi_{1,0}(t)\ dt, \quad \cdots, \quad C_p = \int t^p \phi_{1,0}(t)\ dt.$$
(Hint: Start by showing this for $p=0$, then $p=1$, then generalize.)