GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 6250 Fall 2017
Problem Set #6

Assigned: 11-Mar-17
Due Date: 29-Mar-17

Quiz #2 will be held in class on Wednesday, March 15.

This assignment is due *9:00 AM* in class on Wednesday, March 29.

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

PROBLEM 6.1:
Using you class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

PROBLEM 6.2:
Suppose we have a signal $f(t)$ on $[0, 1]$ which is "bandlimited" in that it only has $N = 2B + 1$ Fourier series coefficients which are non-zero:

$$f(t) = \sum_{k=-B}^{B} \alpha_k e^{j2\pi kt}, \tag{1}$$

for some set of expansion coefficents

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_{-B} \\ \vdots \\ \alpha_0 \\ \vdots \\ \alpha_B \end{bmatrix} \in \mathbb{C}^N.$$

We observe samples M samples of $f(t)$ at locations $t_1, t_2, \ldots, t_M$ which are **not** necessarily uniformly spaced,

$$y[m] = f(t_m), \quad m = 1, \ldots, M. \tag{2}$$

1. Write a MATLAB function `sampmat.m` that takes a vector `smptimes` of length M containing the sample locations and a dimension $N = 2B+1$ (which you can assume is odd), and returns a $M \times N$ matrix $\boldsymbol{A}$ such that when $\boldsymbol{A}$ is applied to a vector of Fourier series coefficients (as in (1)), it returns the sample values in (2).

2. The file `h6problem2.mat` contains vectors `samptimes` and `y` of length $M = 259$, which contain sample times t_m and sample values $f(t_m)$. Find the signal of bandwidth $N = 51$ (so $B = 25$) that best explains these samples in the least-squares sense. Plot your **synthesized estimate** $\hat{f}(t)$ as a function of time.

Last updated 12:19, March 20, 2017

PROBLEM 6.3:
Suppose we have a machine that measures signals $f(t)$ in $L_2([0,1])$ by taking integrals over different regions:

$$y[m] = \int_{a_m}^{b_m} f(t)\ dt, \quad 0 \le a_m \le b_m \le 1, \tag{3}$$

for $m = 1, \ldots, M$. To recover $f(t)$, we model it as being a polynomial of order $N-1$,

$$f(t) = x_{N-1}t^{N-1} + x_{N-2}t^{N-2} + \cdots + x_1 t + x_0. \tag{4}$$

1. Write a MATLAB function `intmat.m` that takes an M vector of lower limits `a`, a M vector of upper limits `b`, and a degree N and returns the $M \times N$ matrix `A` such that if $f(t)$ has the form (4), applying $\boldsymbol{A}$ to[1]

$$\boldsymbol{x} = \begin{bmatrix} x_{N-1} \\ x_{N-2} \\ \cdots \\ x_0 \end{bmatrix}$$

   results in evaluations of the measurements in (3),

$$\boldsymbol{y} = \begin{bmatrix} y[1] \\ y[2] \\ \vdots \\ y[m] \end{bmatrix} = \boldsymbol{A}\boldsymbol{x}.$$

2. The file `hw6problem3.mat` contains vectors `a`, `b`, and `y` of length 20 representing a particular set of lower limits, upper limits, and associated measurements. Find the polynomial of order 9 ($N = 10$) that best describes these measurements in the least-squares sense. Plot your **synthesized estimate** $\hat{f}(t)$ as a function of time — that is, after estimating the coefficients $\hat{x}_{N-1}, \ldots, \hat{x}_0$, form

$$\hat{f}(t) = \hat{x}_{N-1}t^{N-1} + \cdots + \hat{x}_1 t + \hat{x}_0$$

   and plot it as a function of time on $[0,1]$.

[1]The coefficients are ordered in $\boldsymbol{x}$ to match the MATLAB convention for polynomials.

PROBLEM 6.4:
An $N \times N$ *circulant* matrix $\boldsymbol{H}$ has the form

$$\boldsymbol{H} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[2] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[0] \end{bmatrix} \tag{5}$$

1. Show that discrete Fourier vectors

$$\boldsymbol{f}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ e^{j2\pi k/N} \\ e^{j2\pi 2k/N} \\ \vdots \\ e^{j2\pi(N-1)k/N} \end{bmatrix}, \quad k = 0, \ldots, N-1$$

   are eigenvectors for every circulant matrix. What are the eigenvalues?

2. Explain, similar to what we did when discussing eigen-decompositions in the notes, why we can write

$$\boldsymbol{H} = \boldsymbol{F}\boldsymbol{D}\boldsymbol{F}^{\mathrm{H}} \tag{6}$$

   where $\boldsymbol{F}$ is the discrete Fourier matrix whose kth column is $\boldsymbol{f}_k$ above ($\boldsymbol{F}^{\mathrm{H}}$ is the conjugate transpose of $\boldsymbol{F}$), and $\boldsymbol{D}$ is a diagonal matrix. Relate this to the fact that circular convolution in the time domain is multiplication in the DFT domain.

PROBLEM 6.5:

Let

$$\boldsymbol{A} = \begin{bmatrix} 1.01 & 0.99 \\ 0.99 & 0.98 \end{bmatrix}$$

1. Find the eigenvalue decomposition of $\boldsymbol{A}$. Recall that λ is an eigenvalue of $\boldsymbol{A}$ if for some $u[1], u[2]$ (entries of the corresponding eigenvector) we have

$$(1.01 - \lambda)u[1] + 0.99u[2] = 0$$
$$.99u[1] + (0.98 - \lambda)u[2] = 0.$$

   Another way of saying this is that we want the values of λ such that $\boldsymbol{A} - \lambda\mathbf{I}$ (where $\mathbf{I}$ is the 2×2 identity matrix) has a non-trivial null space — there is a nonzero vector $\boldsymbol{u}$ such that $(\boldsymbol{A} - \lambda\mathbf{I})\boldsymbol{u} = 0$. Yet another way of saying this is that we want the values of λ such that $\det(\boldsymbol{A} - \lambda\mathbf{I}) = 0$. Once you have found the two eigenvalues, you can solve the 2×2 systems of equations $\boldsymbol{A}\boldsymbol{u}_1 = \lambda_1\boldsymbol{u}_1$ and $\boldsymbol{A}\boldsymbol{u}_2 = \lambda_2\boldsymbol{u}_2$ for $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

2. If $\boldsymbol{y} = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$, determine the solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$.

3. Now let $y = \begin{bmatrix} 1.1 & 1 \end{bmatrix}^{\mathrm{T}}$ and solve $Ax = y$. Comment on how the solution changed.

4. Suppose we observe

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{e}$$

   with $\|\boldsymbol{e}\|_2 = 1$. We form an estimate $\tilde{\boldsymbol{x}} = \boldsymbol{A}^{-1}\boldsymbol{y}$. Which vector $\boldsymbol{e}$ (over all error vectors with $\|\boldsymbol{e}\|_2 = 1$) yields the maximum error $\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|_2^2$?

5. Which (unit) vector $\boldsymbol{e}$ yields the minimum error?

6. Suppose the components of $\boldsymbol{e}$ are iid Gaussian:

$$e[i] \sim \mathrm{Normal}(0, 1).$$

   What is the mean-square error $\mathrm{E}[\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|_2^2]$?

7. Verify your answer to the previous part in MATLAB by taking $\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$, and then generating $10,000$ different realizations of $\boldsymbol{e}$ using the `randn` command, and then averaging the results. Turn in your code and the results of your computation.

PROBLEM 6.6:

Let $f(\boldsymbol{x})$ be a functional on $\mathbb{R}^N$; that is is, $f(\cdot)$ maps a vector to a real number. Recall the definition of the *gradient* of f at $\boldsymbol{x}$:

$$\nabla_x f = \begin{bmatrix} \frac{df}{dx_1} \\ \frac{df}{dx_2} \\ \vdots \\ \frac{df}{dx_N} \end{bmatrix},$$

where the x_i are the components of $\boldsymbol{x}$.

1. Calculate the gradient for $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_2^2$.
2. Calculate the gradient for $f(\boldsymbol{x}) = \|\boldsymbol{A}\boldsymbol{x}\|_2^2$, where $\boldsymbol{A}$ is an $M \times N$ matrix.
3. Calculate the gradient for $f(\boldsymbol{x}) = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}$.
4. A necessary condition for $\boldsymbol{x}_0$ to be a minimizer of $f(\boldsymbol{x})$ is that $\nabla_{\boldsymbol{x}=\boldsymbol{x}_0} f = \mathbf{0}$. Show that a minimizer $\hat{\boldsymbol{x}}$ of

   $$\min_{\boldsymbol{x}} \; \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2^2$$

   must obey the so-called normal equations

   $$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y}.$$

5. Show that the minimizer $\hat{\boldsymbol{x}}$ of

   $$\min_{\boldsymbol{x}} \; \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2^2 + \delta\|\boldsymbol{x}\|_2^2$$

   is

   $$\hat{\boldsymbol{x}} = \left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \delta\mathbf{I}\right)^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y}.$$

   Make to include an explanation of why $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \delta\mathbf{I}$ is always invertible when $\delta > 0$.

Last updated 12:19, March 20, 2017