GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 6250 Spring 2017
Problem Set #7

Assigned: 20-Mar-17
Due Date: 5-Apr-17

Quiz #3 will be held in class on Wednesday, April 12.

This assignment is due in class on the due date.

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

PROBLEM 7.1:
Using you class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

PROBLEM 7.2:
In this problem, we will explore some ways in which matrices with orthogonal columns preserve norms.

(a) Suppose $\boldsymbol{G}$ is an $N \times N$ matrix with orthonormal columns $\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G} = \mathbf{I}$. Show that $\|\boldsymbol{G}\boldsymbol{x}\|_2^2 = \|\boldsymbol{x}\|_2^2$ for every $\boldsymbol{x} \in \mathbb{R}^N$.

(b) Suppose $\boldsymbol{G}$ is an $N \times R$ matrix, where $R < N$, that has orthonormal columns, so $\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G} = \mathbf{I}$ (but of course in this case $\boldsymbol{G}\boldsymbol{G}^{\mathrm{T}} \neq \mathbf{I}$). Show that

   (a) $\|\boldsymbol{G}\boldsymbol{v}\|_2^2 = \|\boldsymbol{v}\|_2^2$ for all $\boldsymbol{v} \in \mathbb{R}^R$;
   (b) $\|\boldsymbol{G}^{\mathrm{T}}\boldsymbol{x}\|_2^2 \leq \|\boldsymbol{x}\|_2^2$ for all $\boldsymbol{x} \in \mathbb{R}^N$.

(c) Suppose that $\boldsymbol{G}$ is an $N \times N$ matrix with orthonormal columns. Show that $\|\boldsymbol{G}\boldsymbol{A}\|_F^2 = \|\boldsymbol{A}\|_F^2$ for every $N \times K$ matrix $\boldsymbol{A}$. (Recall the definition of the Frobenius norm: $\|\boldsymbol{A}\|_F^2$ is equal to the sum of the squares of the entries of $\boldsymbol{A}$.)

(d) Suppose that $\boldsymbol{A}$ is an arbitrary $M \times N$ matrix. Show that

$$\|\boldsymbol{A}\|_F^2 = \sum_{r=1}^{R} \sigma_r^2,$$

where the σ_r are the singular values of $\boldsymbol{A}$.

PROBLEM 7.3:
Show that if $\boldsymbol{\alpha} \in \mathbb{R}^D$ has independent and normally distributed entries with zero mean and variance σ^2, then for any matrix $\boldsymbol{D}$ with D columns

$$\mathrm{E}[\|\boldsymbol{D}\boldsymbol{\alpha}\|_2^2] = \operatorname{trace}(\boldsymbol{D}^{\mathrm{T}}\boldsymbol{D})\,\sigma^2.$$

PROBLEM 7.4:

(a) Let $\boldsymbol{D}$ be an $R \times R$ diagonal matrix with $D[r,r] \geq 0$. Show that

$$\max_{\substack{\boldsymbol{w}\in\mathbb{R}^R \\ \|\boldsymbol{w}\|_2=1}} \|\boldsymbol{D}\boldsymbol{w}\|_2$$

is maximized by a vector with a 1 in one entry, and zeros in the others. What does the location of that one entry correspond to?

(b) With $\boldsymbol{D}$ as in part(a), describe the minimizer of

$$\min_{\substack{\boldsymbol{w}\in\mathbb{R}^R \\ \|\boldsymbol{w}\|_2=1}} \|\boldsymbol{D}\boldsymbol{w}\|_2$$

(c) Let $\boldsymbol{D}$ be as in part (a), and let $\boldsymbol{U}$ and $\boldsymbol{V}$ be $M \times R$ and $N \times R$ matrices, respectively, with orthonormal columns, $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} = \mathbf{I}$ and $\boldsymbol{V}^{\mathrm{T}}\boldsymbol{V} = \mathbf{I}$. Describe the maximizer and minimizers of

$$\max_{\substack{\boldsymbol{w}\in\mathbb{R}^R \\ \|\boldsymbol{w}\|_2=1}} \|\boldsymbol{U}\boldsymbol{D}\boldsymbol{w}\|_2, \qquad \min_{\substack{\boldsymbol{w}\in\mathbb{R}^R \\ \|\boldsymbol{w}\|_2=1}} \|\boldsymbol{U}\boldsymbol{D}\boldsymbol{w}\|_2$$

and

$$\max_{\substack{\boldsymbol{x}\in\mathbb{R}^N \\ \|\boldsymbol{x}\|_2=1}} \|\boldsymbol{D}\boldsymbol{V}^{\mathrm{T}}\boldsymbol{x}\|_2, \qquad \min_{\substack{\boldsymbol{x}\in\mathbb{R}^N \\ \|\boldsymbol{x}\|_2=1}} \|\boldsymbol{D}\boldsymbol{V}^{\mathrm{T}}\boldsymbol{x}\|_2.$$

(d) Recall the definition of the operator norm of an $M \times N$ matrix $\boldsymbol{A}$:

$$\|\boldsymbol{A}\| = \max_{\substack{\boldsymbol{x}\in\mathbb{R}^N \\ \|\boldsymbol{x}\|_2=1}} \|\boldsymbol{A}\boldsymbol{x}\|_2.$$

Show that for $M \times M$ orthogonal $\boldsymbol{U}$ and $N \times N$ orthogonal $\boldsymbol{V}$, $\|\boldsymbol{U}\boldsymbol{A}\| = \|\boldsymbol{A}\boldsymbol{V}^{\mathrm{T}}\| = \|\boldsymbol{A}\|$.

(e) Suppose that $\boldsymbol{A}$ is an arbitrary $M \times N$ matrix. Show that

$$\|\boldsymbol{A}\| = \max_{1\leq r\leq R} \sigma_r,$$

where the σ_r are the singular values of $\boldsymbol{A}$.

PROBLEM 7.5:

Let $\boldsymbol{h} \in \mathbb{R}^{1024}$ be defined by

$$h[n] = \begin{cases} 1 & 0 \le n \le 31 \\ 0 & 32 \le n \le 1023 \end{cases},$$

and let $\boldsymbol{H}$ be the circulant matrix generated by $\boldsymbol{h}$:

$$\boldsymbol{H} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[2] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[0] \end{bmatrix}.$$

(Here $N = 1024$.) As we saw in the previous homework, we can write

$$\boldsymbol{H} = \boldsymbol{F}\boldsymbol{D}\boldsymbol{F}^{\mathrm{H}} = \sum_{k=0}^{N-1} d[k] \boldsymbol{f}_k \boldsymbol{f}_k^H$$

where the matrix $\boldsymbol{F}$ has columns

$$\boldsymbol{f}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ e^{j2\pi k/N} \\ e^{j2\pi 2k/N} \\ \vdots \\ e^{j2\pi(N-1)k/N} \end{bmatrix}, \quad k = 0, \ldots, N-1,$$

and the $d[k]$, which are along the diagonal of $\boldsymbol{D}$, are the DFT coefficients of $\boldsymbol{h}$. This is not quite an SVD of $\boldsymbol{H}$ (since $\boldsymbol{H}$ is real-values, and both $\boldsymbol{F}$ and $\boldsymbol{D}$ are complex-valued), but it has the same essential properties — just be careful below about conjugates and the difference between z^2 and $|z|^2$.

1. In MATLAB, construct $\boldsymbol{H}$. If you like, you can use the diagonalization formula above to do this; it is easy to construct $\boldsymbol{F}$ using `F = 1/sqrt(N)*conj(fft(eye(N)));`. Turn in your code and a plot of `imagesc(H)`.

2. Suppose we observe
$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{e},$$
and attempt to recover $\boldsymbol{x}$ using Tikhonov regularization,
$$\min_{\boldsymbol{x}} \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|_2^2 + \delta \|\boldsymbol{x}\|_2^2.$$
Show that we can write $\hat{\boldsymbol{x}} = \boldsymbol{G}\boldsymbol{y}$, where $\boldsymbol{G}$ is also a circulant matrix. What are the Fourier coefficients for the $\boldsymbol{g} \in \mathbb{R}^N$ whose circular shifts generate $\boldsymbol{G}$?

3. The file `hw7problem5.mat` contains a vector $\boldsymbol{x}$ of length 1024 and a noisy observation through $\boldsymbol{H}$, $\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{e}$, where $e[n] \sim \mathrm{Normal}(0, 1)$. Estimate $\boldsymbol{x}$ using Tikhonov regulatization for $\delta = 10^{-4}, 10^{-2}, 1$ and 5. For each value of δ, plot $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ overlaid on one another. Comment on how the estimate changes as δ gets larger. Turn in your plots and your code.