Kalman Filters

Problem 2: A process $s(n)$ is given by

$$s(n) = w(n-1) - w(n-2)$$

where $w(n)$ is a zero-mean white process with variance $\sigma_w^2 = 2$. Measurements $y(n)$ of $s(n)$ are given by

$$y(n) = s(n) + v(n)$$

where $v(n)$ is zero-mean white noise with a variance $\sigma_v^2 = 1$, and $v(n)$ is independent of $w(n)$. Determine the equations for a Kalman filter that produces an estimate of $d(n)$ at time n based on the measurements $y(i)$ for $i = 1, 2, \ldots, n$. Simplify them as much as you can, i.e., do not simply write down the Kalman filter equations. **Note:** There are (at least) two ways to set up the state and measurement equations for this problem. If you do not want to use the noise $w(n)$ as part of the state vector, then you may consider writing the state and measurment equations in the following form:

$$\begin{aligned} \mathbf{x}(n+1) &= \mathbf{A}\mathbf{x}(n) + \mathbf{B}w(n) \\ s(n) &= \mathbf{C}\mathbf{x}(n) \\ y(n) &= s(n) + v(n) \end{aligned}$$

Problem 3: An autoregressive process of order one is described by the difference equation

$$x(n+1) = 0.5x(n) + w(n)$$

where $w(n)$ is zero-mean white noise with a variance $\sigma_w^2 = 0.64$. The observed process $y(n)$ is described by

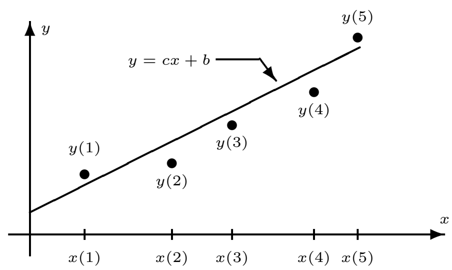
$$y(n) = x(n) + v(n)$$

where $v(n)$ is zero-mean white noise with a variance $\sigma_v^2 = 1$. A Kalman filter is designed and implemented to estimate $x(n)$ from the observations $y(n)$. The initial conditions are $\hat{x}(0|0) = 0$ and $E\{\epsilon^2(0|0)\} = 1$ where $\epsilon(0|0) = x(0) - \hat{x}(0|0)$. The steady state Kalman solution for the estimate of $x(n)$ has the form

$$\widehat{x}(n+1|n+1) = a\widehat{x}(n|n) + by(n)$$

Find the values for a and b.

Problem 5: Suppose that we have a sequence of data points, $x(n), y(n)$, as illustrated in the figure below, where $x(1) < x(2) < \cdots < x(n)$.



and that we would like to find the best fit of a line to this data,

$$y = mx + b$$

To do this, we may consider the values $y(i)$ to be noisy samples of the exact linear relationship $y = mx + b$, and use a Kalman filter to recursively find the minimum mean-square estimate, $\widehat{m}$ and $\widehat{b}$, of m and b.

(a) Assuming that the measurements $x(n)$ are *exact*, and that the values of $y(n)$ are corrupted by zero mean white noise $v(n)$ with a variance $\sigma_v^2 = 1$, set up the state and observation equations for this Kalman filtering problem (i.e., how would you define the state vector $\mathbf{z}(n)$, what is the state transition matrix $\mathbf{A}(n+1, n)$, the matrix $\mathbf{C}(n)$, and so on).[3] For the initial conditions, assume that there is no prior knowledge of m and b.

(b) Find the estimates $\widehat{m}(1|1)$ and $\widehat{b}(1|1)$ of m and b if the first measurement is

$$y(1) = 1 \quad ; \quad x(1) = 1$$

(c) To what does $\mathbf{P}(n)$ converge as $n \to \infty$? How does this compare to what you would find, in practice, for the variance of the state estimation error?

[3] Note that I am suggesting that you use $\mathbf{z}(n)$ for the state vector, since $x(n)$ is used for the x-coordinate of the data.

Problem 6: Problem 2: Consider a system consisting of two sensors, each making a single measurement of an unknown constant x. Each measurement is noisy and may be modeled as follows

$$\begin{aligned} y(1) &= x + v(1) \\ y(2) &= x + v(2) \end{aligned}$$

where $v(1)$ and $v(2)$ are zero mean uncorrelated random variables with variance σ_1^2 and σ_2^2, respectively. In the absence of any other information, we seek the best linear estimate of x of the form

$$\widehat{x} = k_1 y(1) + k_2 y(2)$$

(a) What constraints must be placed on k_1 and k_2 to guarantee that the estimate $\widehat{x}$ is unbiased, i.e., that $E\{x - \widehat{x}\} = 0$?

(b) If $\sigma_1^2 = \sigma_2^2$, what are the values for k_1 and k_2 that minimize the mean square error

$$\xi = E\big\{(x - \widehat{x})^2\big\}$$

Hint: No complex derivations are necessary in order to answer this question.

(c) Repeat part (a) for the case in which $\sigma_2^2 \to \infty$. Again, no complex derivations are necessary to answer this question.

(d) Set this problem up within the framework of Kalman filtering, treating the measurements $y(1)$ and $y(2)$ sequentially, and find the values for k_1 and k_2 for the general case in which $\sigma_1^2 \neq \sigma_2^2$.

Problem 8: An aircraft tracking system consists of two radars making measurements of distance and velocity. The first radar is designed to give accurate distance measurements (to within 1 km) at the expense of poor velocity measurements (variance of 5 km/h). The second gives accurate velocity measurements, to within 1 km/h, and poor distance measurements (variance of only 10 km). An aircraft makes radio contact and reports a distance of 250 km (variance of 2 km), and a velocity of 1200 km/h (variance of 50 km/h). Radar contact is immediately established and provides the following measurements:

1. Radar 1 (accurate distance): Distance = 260 km, velocity = 1100 km/h
2. Radar 2 (accurate velocity): Distance = 270 km, velocity = 1300 km/h

Find the minimum mean square error estimate of the aircraft's distance.

Problem 9: In many cases, the error covariance matrix $\mathbf{P}(n|n-1)$ will converge to a steady-state value $\mathbf{P}$ as $n \to \infty$. Assume that $\mathbf{C}$, $\mathbf{Q}_w$, and $\mathbf{Q}_v$ are the limiting values of $\mathbf{C}(n)$, $\mathbf{Q}_w(n)$, and $\mathbf{Q}_v(n)$, respectively.

(a) For $\mathbf{A}(n) = \mathbf{I}$, show that if $\mathbf{P}(n|n-1)$ converges to a steady state value $\mathbf{P}$, then the limiting value satisfies the *algebraic Ricatti equation*

$$\mathbf{P}\mathbf{C}^H\left(\mathbf{C}\mathbf{P}\mathbf{C}^H + \mathbf{Q}_v\right)^{-1}\mathbf{C}^H\mathbf{P} - \mathbf{Q}_w = \mathbf{0}$$

(b) Derive the Ricatti equation for a general state transition matrix $\mathbf{A}(n)$ that has a limiting value of $\mathbf{A}$.