Lecture 1 - Sampling

Wednesday, January 11, 2017 8:59 AM

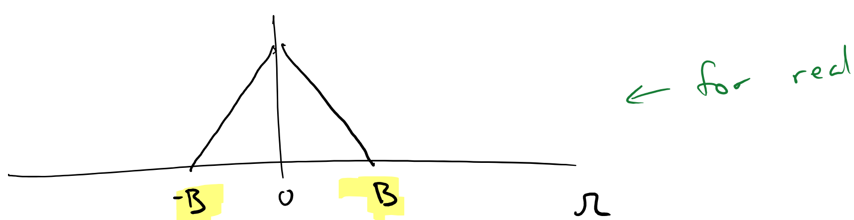
Fundamental theorem of DSP

If $x_c(t)$ is bandlimited to B ($X_c(j\Omega) = 0$ for $|\Omega| \geq B$), then it can be perfectly reconstructed from samples spaced $T \leq \pi/B$ apart.

$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n]\, h_T(t - nT), \quad x[n] = x(nT)$$

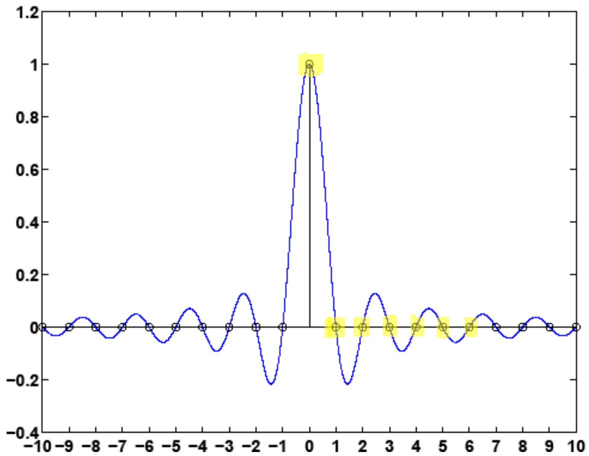
$$h_T(t) = \frac{\sin(\pi t/T)}{\pi t/T} \quad \text{"sinc" function}$$

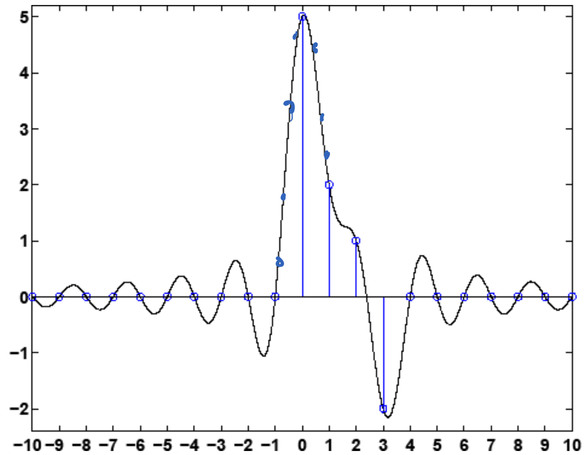
The Shannon-Nyquist sampling theorem. →
if we sample a signal at a rate more than double its highest frequency, we can perfectly reconstruct that signal.

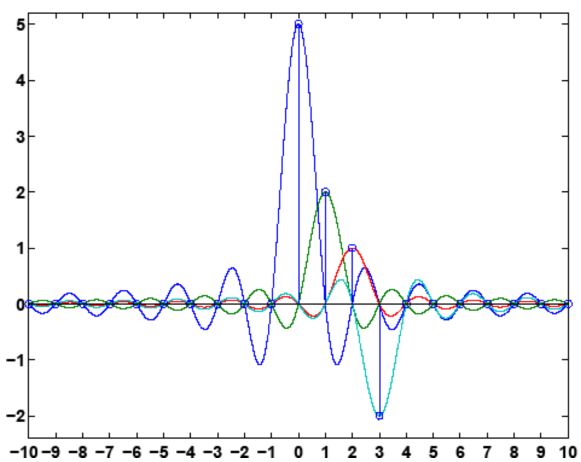


This also works for signals with band-limited spectra not centered at zero...

reconstruction is with a modulated sinc function







Frequency-Domain interpretation

There are multiple Fourier transforms ...

Fourier Transform $x_c(t) \leftrightarrow X(j\Omega)$

operates on continuous, non-periodi signals

$X(j\Omega)$ is continuous, non-periodic

Fourier Series $x_p(t) \leftrightarrow \{a_k\}$

$x_p(t)$ is continuous & periodic

$\{a_k\}$ is discrete & non-periodic

The DTFT is used as an analysis tool

Discrete-Time Fourier Transform (DTFT)

$x[n] \leftrightarrow X(e^{j\omega})$

$x[n]$ is discrete, & non-periodic

$X(e^{j\omega})$ is continuous & periodic

in DSP we usually use the DFT

Discrete Fourier Transform (DFT)

$x[n] \leftrightarrow X[k]$

DFT $\longrightarrow$ Discrete Fourier Transform (DFT)

$x[n] \longleftrightarrow X[k]$

If we have a finite length sequence, The DFT is a sampled DTFT

$x[n]$ is discrete & periodic

$X[k]$ is discrete & periodic

Let's find the DTFT of $x[n]$ that is a sampled version of $x_c(t)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_c(n\underset{\text{sample period}}{T})\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(j\Omega)\, e^{j\Omega n T}\, d\Omega \right)}_{\text{inverse F.T. of } X_c(j\Omega) \longleftrightarrow x_c(t)} e^{-j\omega n}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(j\Omega) \cdot \underbrace{\left(\sum_{n=-\infty}^{\infty} e^{jn(\Omega T - \omega)} \right)}_{?} d\Omega$$

Poisson Summation formula

$$\sum_{n=-\infty}^{\infty} e^{jn\omega} = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

Dirac delta

$$= \int_{-\infty}^{\infty} X_c(j\Omega) \cdot \sum_{k=-\infty}^{\infty} \delta(\Omega T - \omega - 2\pi k)\, d\Omega$$

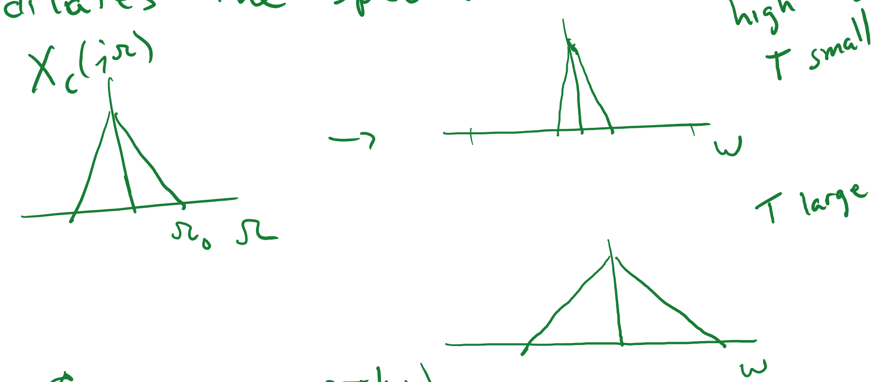
$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} X_c(j\Omega)\, \delta(\Omega T - \omega - 2\pi k)\, d\Omega$$

$k=-\infty \quad -\infty$

$$\boxed{X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega+2\pi k}{T}\right)\right)}$$

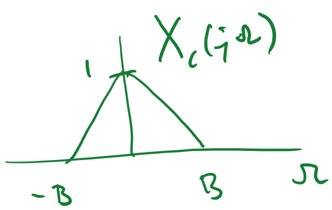
① When $k=0$ $\quad X(e^{j\omega}) \to \frac{1}{T} X_c\left(j\frac{\omega}{T}\right)$

dilates the spectrum

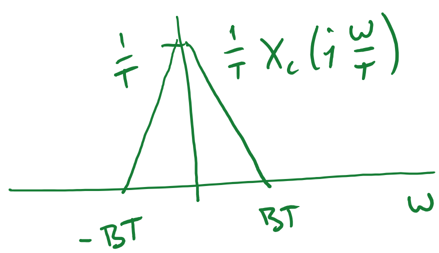


② $\frac{1}{T} X_c\left(j\frac{\omega}{T}\right) \to \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega+2\pi k}{T}\right)\right)$

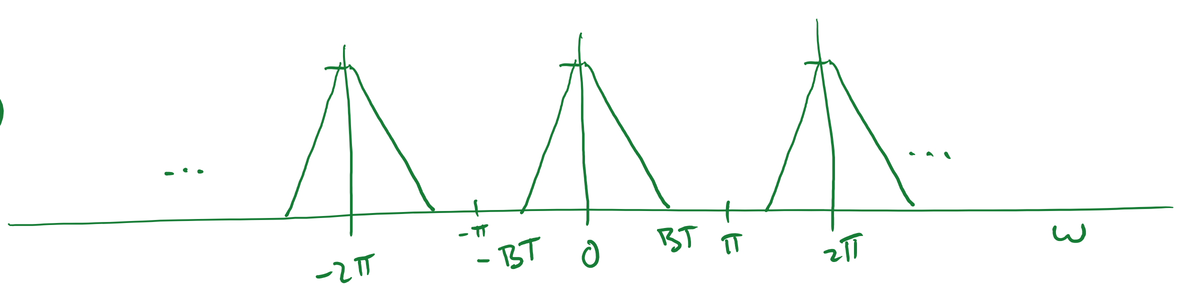
└ This makes the dilated spectrum periodic



①

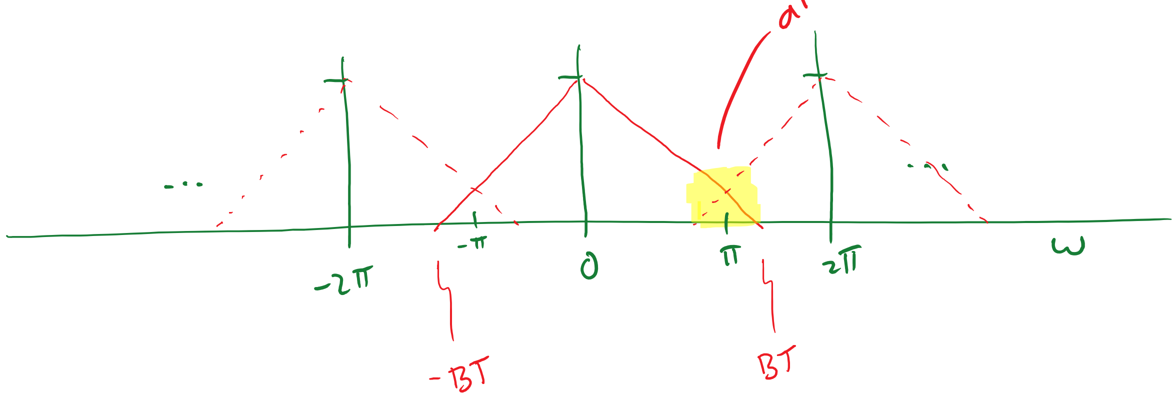


②



if $T > \pi/B$ what happens?

Then $BT > B\left(\frac{\pi}{B}\right) = \pi$



13 January 2017

Reconstruction

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\, h_T(t - nT)$$

$\Downarrow$ F.T. w.r.t. T

$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, H_T(j\Omega)\, e^{j\Omega nT}$$

$$= H_T(j\Omega) \sum_{n=-\infty}^{\infty} x[n]\, e^{j\Omega nT}$$

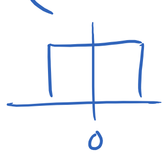
DTFT

$$= H_T(j\Omega)\, X(e^{j\Omega T})$$

$\Omega T \leftrightarrow \omega$

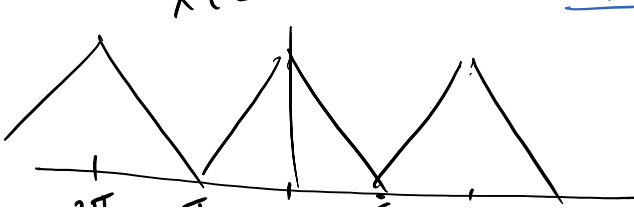
F.T. D.T.F.T

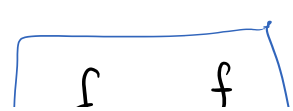
ΩT dilates the periodic spectrum

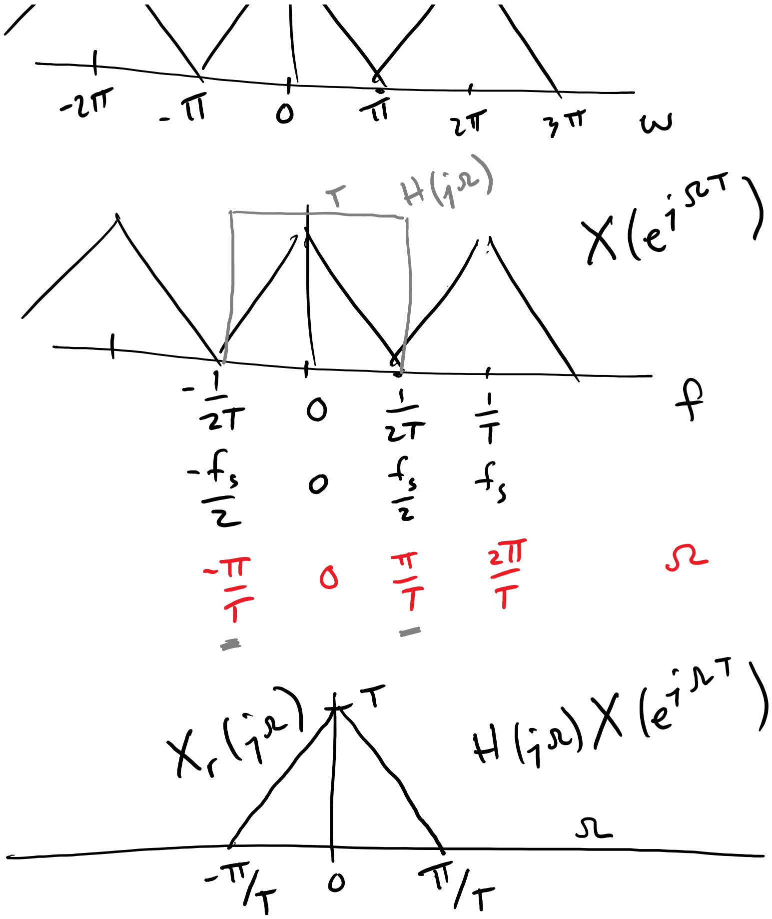


$\leftarrow$ restrict $X_r(j\Omega)$ to the fundamental period of $X(e^{j\Omega T})$

$X(e^{j\omega})$







$$\boxed{\frac{f_s}{2\pi} = \frac{f}{\omega}}$$

$$f_s = \frac{1}{T}$$

$$\frac{1/T}{2\pi} = \frac{f}{\omega}$$

$$f = \frac{\omega}{T2\pi}$$

$$\Omega = 2\pi f$$