Lecture 2 (cont.) - Friday, Jan 13 - Monday, Jan 16

Saturday, January 14, 2017 11:18 PM

Basis Expansions

We will be using the following key concepts from linear algebra

- norms
- bases / change of bases
- inner products / orthogonality / projections
- linear subspaces
- linear operators (matrices in finite dimensions)
- eigenvalues / singular

Example: Fourier series

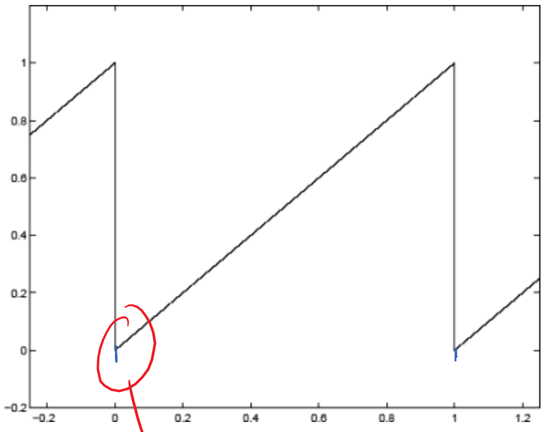
↪ periodic sequence ⟶ superposition of harmonic sinusoids

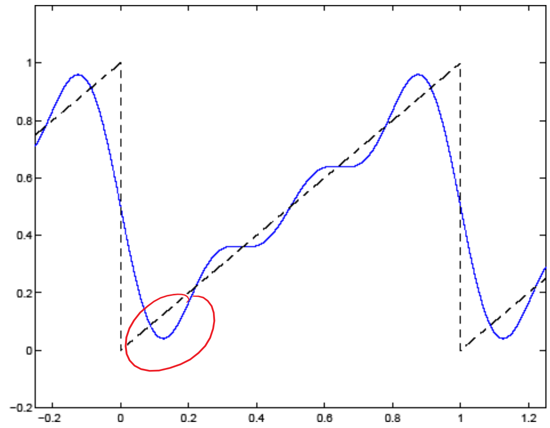
$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \underline{\underline{e^{j2\pi kt/T}}}$$

⟵ synthesize a signal from a list of numbers

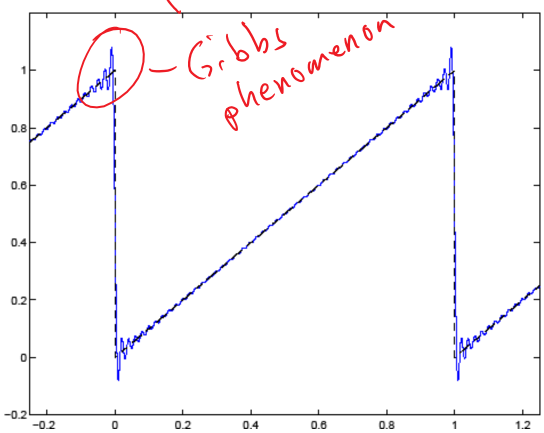
where $$\alpha_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi kt/T}\, dt$$

maps a signal to a discrete list of numbers





$|k| \le 3$



Gibbs phenomenon

$|k| \le 50$

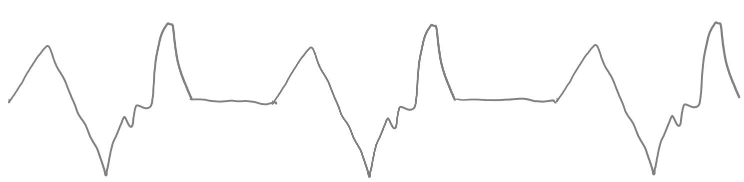
This is mathematically equivalent to the sampling theorem

Suppose that $x(t)$ is zero outside of $[-T/2, T/2]$

Then $$X(j\Omega) = \int_{-T/2}^{T/2} x(t)\, e^{-j\Omega t}\, dt$$

$$\alpha_k = \frac{1}{T} X\left(j \frac{2\pi k}{T}\right)$$ ← F. Series coefficients are just samples of the Fourier transform!

← critically sampled in freq.



← oversampled in freq.

We can write the Fourier Transform as a combination of samples of α_k

$$X(j\Omega) = \int_{-T/2}^{T/2} \underbrace{\sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi kt/T}}_{x(t)} e^{-j\Omega t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} \alpha_k \int_{-T/2}^{T/2} e^{j(2\pi k/T - \Omega)t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} \alpha_k \frac{2T \sin(\Omega T/2 - \pi k)}{\Omega T - 2\pi k}$$

$$X(j\Omega) = \sum_{k=-\infty}^{\infty} X\left(j\frac{2\pi k}{T}\right) h_{2\pi/T} \underbrace{(\Omega - 2\pi k/T)}_{\text{sinc}}$$

The Fourier transform of a signal which is time-limited to T can be reconstructed from samples taken $2\pi/T$ in frequency.

Example #2: Polynomials

obvious - any m^{th}-order polynomial as a superposition of functions, $1, t, t^2, \ldots, t^m$

more generally, analytic functions can be re-written as "indefinite degree" polynomials using a Taylor series

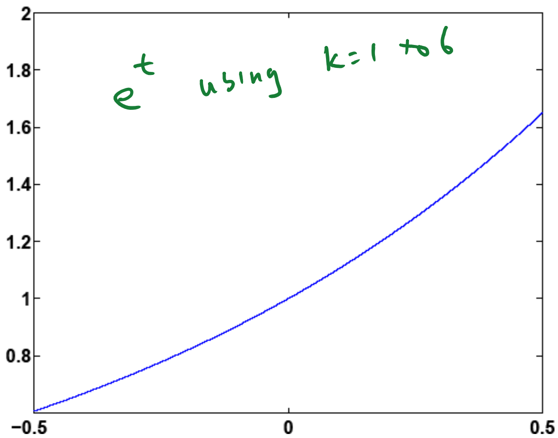
e.g. on $[-1/2, 1/2]$, $e^t = \sum_{k=0}^{\infty} \alpha_k t^k$, where $\alpha_k = \frac{1}{k!}$

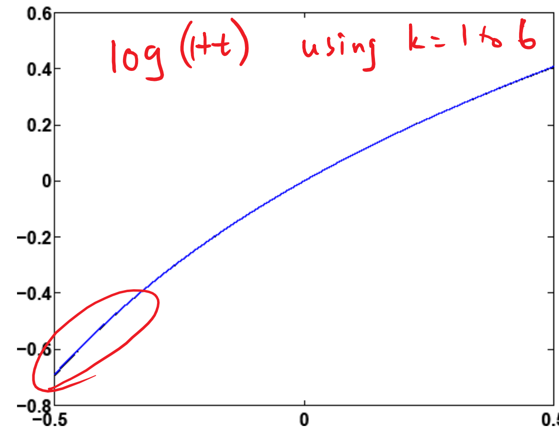
or, $\log(1+t) = \sum_{k=0}^{\infty} \alpha_k t^k$

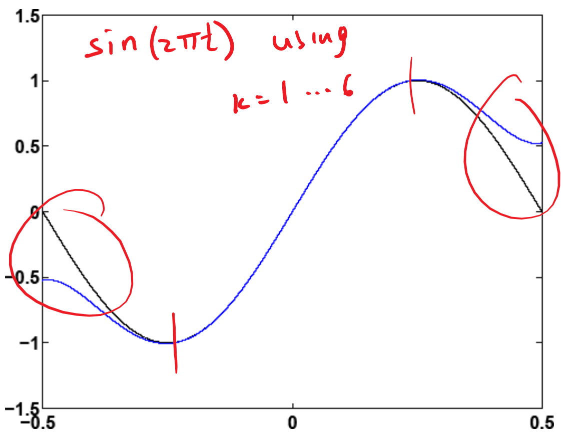
where $\alpha_k = \frac{(-1)^{k+1}}{k}$

or $\sin(2\pi t) = \sum_{k=0}^{\infty} \alpha_k t^k$

where $\alpha_k = \begin{cases} \dfrac{(-1)^{(k+3)/2} (2\pi)^{k+1}}{(k+1)!} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$







Aside

$\cos(\omega_0 n) = \mathrm{Re}\{e^{j\omega_0 n}\}$

$= \mathrm{Re}\{e^{j\omega_0 (n-1)} e^{j\omega_0}\}$

we have this from last time

complex number (pre-compute)

start with

$W_0 = \cos(\omega_0) + i \sin(\omega_0)$

```
p = 1
n = 0
repeat
    cos(w0 n) = real(p);
    p = p * W0
    n = n+1
end repeat
```

Example #3 - splines

... next time

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Splines

We can interpolate data with polynomial splines

Given a set of locations $t_1, t_2, \ldots, t_k$
and function values at those locations
$v_{t_1}, v_{t_2}, \ldots, v_{t_k}$

An ℓ^{th}-order spline is a function $x(t)$ which obeys

$$x(t_k) = v_{t_k} \quad \text{for } k = 1, \ldots, k$$

– and –

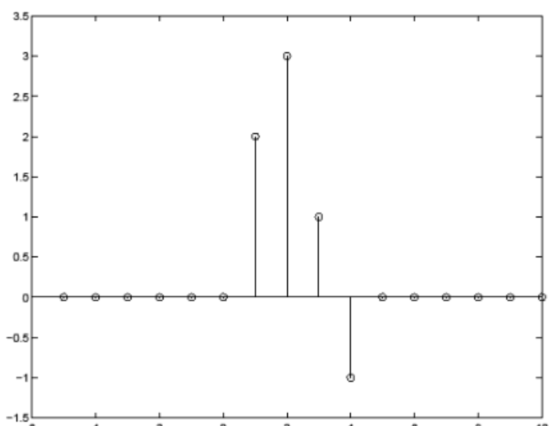
$x(t)$ is an ℓ^{th} order polynomial between the t_k's

If $\ell \geq 1$, the spline function is continuous and will have $\ell - 1$ derivatives which are continous at each t_k.

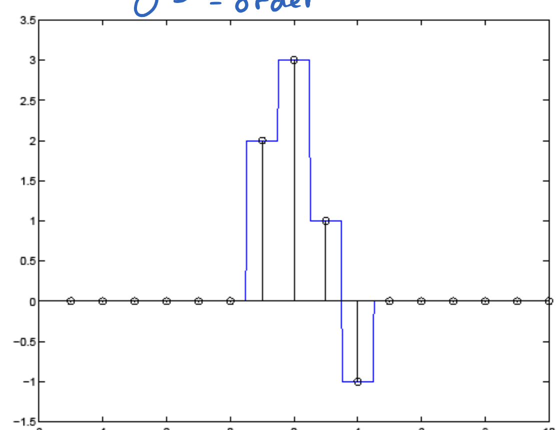
Example: $t_1 = 1,\ t_2 = 2,\ t_3 = 3,\ t_4 = 4$ ← sampling at integers

$v_1 = 2,\ v_2 = 3,\ v_3 = 1,\ v_4 = -1$

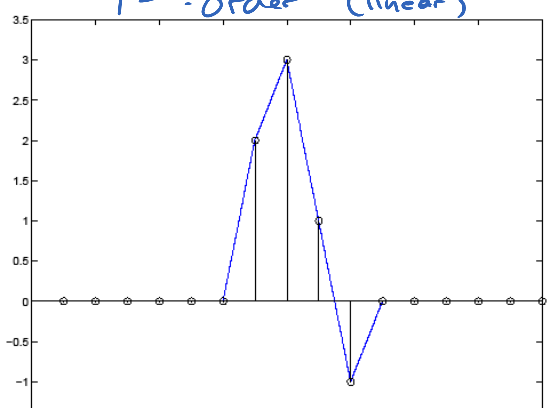
$v_{t_k} = 0$ for all other k



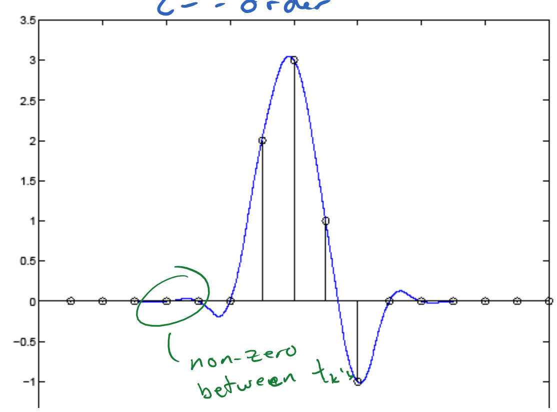
0^{th}-order

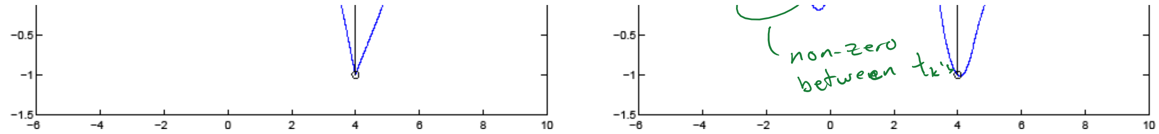


1^{st}-order (linear)



2^{nd}-order





$$x_0(t) = \sum_{k=1}^{4} \alpha_k \, b_0(t-k) \quad \text{where} \quad b_0 = \begin{cases} 1 & -1/2 \le t < 1/2 \\ 0 & \text{else} \end{cases}$$

(for integer t_k's)

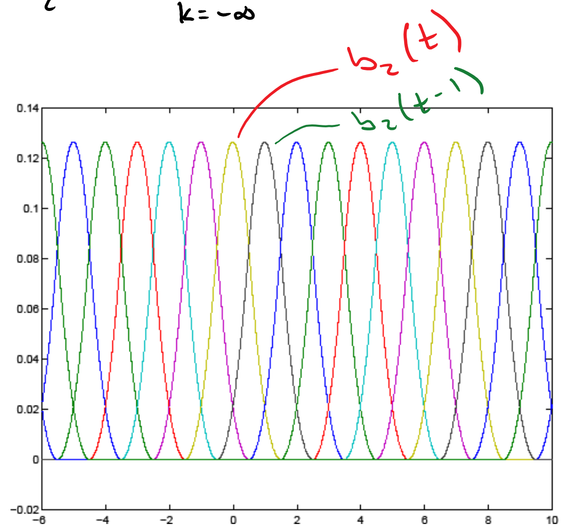
$\alpha_1 = 2, \quad \alpha_2 = 3, \quad \alpha_3 = 1, \quad \alpha_4 = -1$

$$x_1(t) = \sum_{k=1}^{4} \alpha_k \, b_1(t-k) \quad \text{where} \quad b_1(t) = \begin{cases} t+1 & -1 \le t \le 0 \\ 1-t & 0 < t \le 1 \\ 0 & \text{else} \end{cases}$$

$\alpha_1 = 2, \quad \alpha_2 = 3, \quad \alpha_3 = 1, \quad \alpha_4 = -1$

$b_1(t) = (b_0 * b_0)(t)$

$$x_2(t) = \sum_{k=-\infty}^{\infty} \alpha_k \, b_2(t-k) \quad \text{where} \quad b_2(t) = \begin{cases} (t+3/2)^2/2 & -\frac{3}{2} \le t \le -1/2 \\ -t^2 + 3/4 & -\frac{1}{2} \le t \le 1/2 \\ (t-3/2)^2/2 & 1/2 \le t \le 3/2 \\ 0 & |t| \geq 3/2 \end{cases}$$



$b_2(t) = (b_1 * b_0)(t)$
$= (b_0 * b_0 * b_0)(t)$

$b_2(t)$'s overlap
$\Downarrow$
$\therefore$ the α_k's are all non-zero and there are an infinite # of terms.

B-splines
"B" for basis

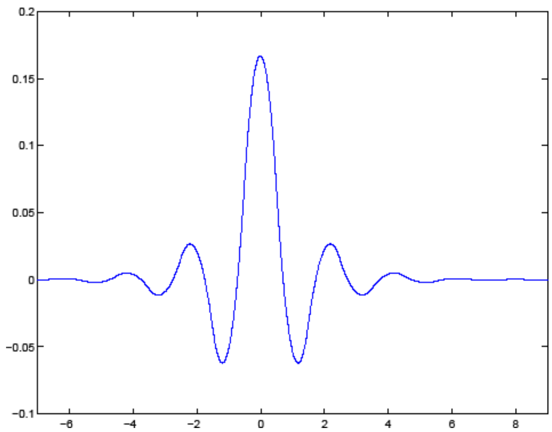
$$x_\ell(t) = \sum_{k=-\infty}^{\infty} \alpha_k \, b_\ell(t-k) \qquad b_\ell(t) = \underbrace{b_0(t) * \cdots * b_0(t)}_{\ell \text{ times}}$$

So any ℓ^{th}-order polynomial spline $x_\ell(t)$ is

uniquely represented by a list of numbers
$\{\alpha_k, k \in \mathbb{Z}\}$

$$\alpha_k = \int_{-\infty}^{\infty} x_\ell(t)\, \underbrace{\tilde{b}_\ell(t-k)}\, dt$$

complementary function $\tilde{b}_\ell(t)$



for $\ell=2$, kind of like a sinc function

We can, in many circumstances, represent continuous signals as lists (possibly infinite) of numbers.

↳ we can use linear algebra!