Vector Spaces

Monday, January 16, 2017 9:42 AM

Linear Algebra has become as basic and as applicable as calculus, and fortunately it is easier.
- Gilbert Strang

A vector space S is composed of
a set of elements called vectors
and members of a field $\mathbb{F}$ called scalars.
(a set of numbers for which multiplication and addition are defined.

The space also has rules.

vector addition → combines two vectors and creates a third (denoted by '+')

scalar multiplication → combines a vector & a scalar to get another vector

The '+' operation the following four rules
for all $\mathbf{x}$ & $\mathbf{y} \in S$

1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
2. $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$
3. There is a unique zero vector $\mathbf{0}$ such that
$$\mathbf{x} + \mathbf{0} = \mathbf{x} \quad \forall \mathbf{x} \in S$$
4. For each vector $\mathbf{x} \in S$, there is a unique vector called $-\mathbf{x}$ such that
$$\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$$

Scalar multiplication must obey the following four rules for all $a, b \in \mathbb{F}$ and $\mathbf{x}, \mathbf{y} \in S$

1. $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$ (distributive)

   $(a+b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$

2. $(ab)\mathbf{x} = a(b\mathbf{x})$ (associative)

3. $\underbrace{1}_{\text{multiplicative identity}} \cdot \mathbf{x} = \mathbf{x}$

4. For the additive identity of $\mathbb{F}$, which we write as 0,

$$\underbrace{0}_{\text{scalar}} \cdot \mathbf{x} = \underbrace{\mathbf{0}}_{\text{vector}}$$

S is closed under scalar multiplication and vector addition

$$\mathbf{x}, \mathbf{y} \in S \Rightarrow a\mathbf{x} + b\mathbf{y} \in S \quad \forall a, b \in \mathbb{F}$$

why we often call vector spaces "linear vector spaces"

Examples

- $\mathbb{R}^N$ $\quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$ where x_i are real

  we use obvious + & multiplication

- $\mathbb{C}^N$ same as $\mathbb{R}^N$ except x_i are complex

- Bounded, continuous functions, $f(t)$ on the interval $[a, b]$ that are real-valued.

vector addition = pointwise $f_1(t) + f_2(t)$

scalar multiplication = multiplying by $a \in \mathbb{R}$ pointwise

$a < \infty$

- $GF(2)^N$

  Scalar field is $\{0,1\}$, vectors are lists of N bits.

  Addition is modulo 2 so

  $$0+0=0$$
  $$0+1=1 \qquad 1+0=1$$
  $$1+1=0$$

- Bounded, continuous functions $f(t)$ on $[a,b]$ such that $|f(t)| \leq 1$

  linear vector space? – not closed under addition

  All allowed operations must result in a vector that is still in S.

18 Jan 2017

Linear Subspaces

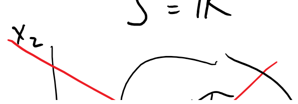
A (non-empty) subset T of S is called a linear subspace of S if $\forall a,b \in \mathbb{F},\ \mathbf{x},\mathbf{y} \in T \Rightarrow a\mathbf{x} + b\mathbf{y} \in T$

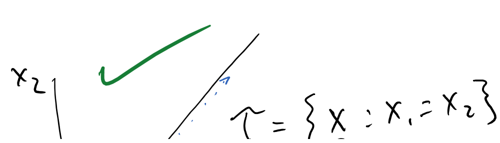
It must be true that

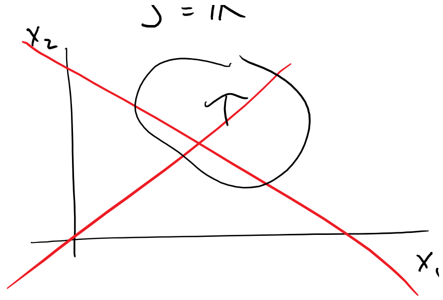
$$\mathbf{0} \in T$$

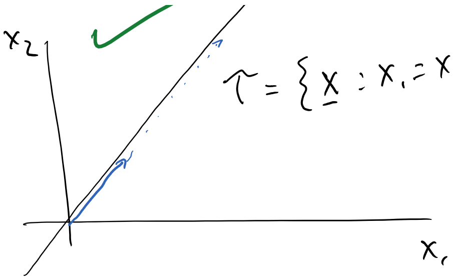
Note, T is a linear vector space by itself.

$S = \mathbb{R}^2$









$T = \{ \underline{x} : x_1 = x_2 \}$

$c \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$

$\begin{pmatrix} cx_1 \\ cx_1 \end{pmatrix}$

more examples

1) $S = \mathbb{R}^5$

$T = \{ \underline{x} : x_4 = 0, x_5 = 0 \}$ yes

2) $S = \mathbb{R}^5$

$T = \{ \underline{x} : x_4 = 1, x_5 = 1 \}$ no

3) $S = C([0,1])$ (bounded, continuous function $[0,1]$)

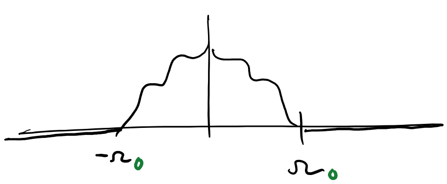
$T = \{ \text{polynomials of degree } p \}$

$\alpha_p t^p + \alpha_{p-1} t^{p-1} + \ldots + \alpha_0$ $\qquad \sum_{k=0}^{p} \alpha_k t^k$

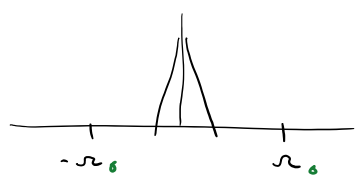
$\beta_p t^p + \ldots + \beta_0$

4) $S = C(\mathbb{R})$

$T = \{ f(t) : f \text{ is bandlimited to } \Omega_0 \}$



+



$a X(\Omega) + b Y(\Omega) = Z(\Omega)$

← cannot be zero where X & Y are zero

5) $S = \mathbb{R}^N$ $\qquad N > 5$

$T = \{ \underline{x} : \underline{x}$ has no more than 5 non-zero components$\}$

$$\begin{pmatrix} a \\ b \\ 0 \\ 0 \\ c \\ d \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ e \\ f \\ 0 \\ 0 \\ g \end{pmatrix} = \begin{pmatrix} a \\ b \\ e \\ f \\ c \\ d \\ g \end{pmatrix}$$

6) $S = \mathbb{R}^N$

$T = \{ \underline{x} : \langle \underline{x}, \underline{c} \rangle = 3 \}$, $\underline{c} \in \mathbb{R}^N$ is a fixed vector and $\langle \underline{x}, \underline{c} \rangle = \underline{c}^T \underline{x}$ is an inner product.

7) $S = C([0,1])$

$T = \{ f(t) : f(t) = a\cos(2\pi t) + b\sin(2\pi t)$ for $a, b \in \mathbb{R} \}$

Linear Combinations and spans

Let $M = \{ \underline{v}_1, \ldots, \underline{v}_N \}$ where $\underline{v}_i \in S$ (linear vector space)

Define: A **linear combination** of vectors in M is a sum of the form

$$a_1 \underline{v}_1 + a_2 \underline{v}_2 + \cdots + a_N \underline{v}_N$$

for some $a_1, \ldots, a_N \in \mathbb{F}$

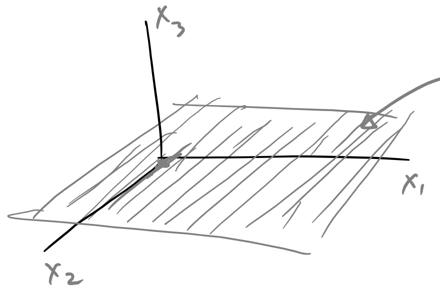
Define: The **span** of M is the set of all linear combinations of M

$$\text{span}(M) = \text{span}(\{ \underline{v}_1, \ldots, \underline{v}_N \})$$

Example:

$S = \mathbb{R}^3$, $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\text{span}(\{\underline{v}_1, \underline{v}_2\}) = (x_1, x_2) \text{ plane}$$



span = (x_1, x_2) plane

$$\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

for some $a, b \in \mathbb{R}$

for a given x_1, x_2, $a = x_1$

$b = x_2 - x_1$

Q: $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\quad \text{span}(\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}) = (x_1, x_2) \text{ plane}$

$\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ $\quad \text{span}(\dots) = ?\ \mathbb{R}^3$

$$\det \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$$

Example: $S = \{ x(t) : x(t) \text{ is periodic with period } 2\pi \}$

$$M = \{ e^{jkt} \}_{k=-B}^{B}$$

The $\text{span}(M)$ = periodic, bandlimited (to B) functions, i.e.

$$x(t) = \sum_{k=-B}^{B} c_k e^{jkt}$$

Linear Dependence

Definition: A set of vectors $\{\underline{v}_i\}_{i=1}^{N}$ is said to be

Linear Dep

Definition: A set of vectors $\{\underline{v}_j\}_{j=1}^{N}$ is said to be linearly dependent if there exist scalars $a_1, \ldots, a_N$ not all $= 0$, such that

$$\sum_{n=1}^{N} a_n \underline{v}_n = \mathbf{0}$$

Definition: If $\sum_n a_n \underline{v}_n = \underline{0}$ only when all $a_j = 0$ then $\{\underline{v}_n\}_{n=1}^{N}$ is said to be linearly independent.

example:

$$S = \mathbb{R}^3, \quad \underline{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

find a_1, a_2, a_3 so that

$$a_1 \underline{v}_1 + a_2 \underline{v}_2 + a_3 \underline{v}_3 = \underline{0}$$

$$a_1 = a_3 = 1, \quad a_2 = -3$$

note, for these, $\mathrm{span}(\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}) = \mathrm{span}(\{\underline{v}_1, \underline{v}_2\})$
$= \mathrm{span}(\{\underline{v}_2, \underline{v}_3\}) = \mathrm{span}(\{\underline{v}_1, \underline{v}_3\})$

Suppose that $\{\underline{v}_1, \ldots, \underline{v}_N\}$ are linearly dependent

then $\sum_n a_n \underline{v}_n = \underline{0} \Rightarrow \underline{v}_k = \frac{-1}{a_k} \sum_{n \neq k} a_n \underline{v}_n$

(if $a_k \neq 0$)

Therefore, we can remove at least one vector without changing the span. Repeat until the remaining set is linearly independent.

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$$

$$-\frac{1}{a_3}(a_1 v_1 + a_2 v_2) = \cancel{a_3} v_3$$

23 January 2017

Bases

Def. A basis of a linear vector space S is a (countable) set of vectors $\mathbb{B}$ such that

1. $\operatorname{span}(\mathbb{B}) = S$
2. $\mathbb{B}$ is linearly independent

The second condition ensures that all bases of S will have the same number of elements (possibly infinite).

The dimension of S is the number of elements required in a basis for S.

Examples

- $\mathbb{R}^N$ with $$\underbrace{\mathbb{B} = \{\underline{v}_1, \underline{v}_2, \dots, v_N\}}_{\text{standard basis for } \mathbb{R}^N} = \left\{ \overset{e_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} \overset{e_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \dots, \overset{e_N}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}} \right\}$$
- $\mathbb{R}^N$ with any N linearly-independent vectors
- $S = \{\text{polynomials of degree at most } p\}$

  $\mathbb{B} = \{1, t, t^2, \dots, t^p\}$

  dimension of $S = p+1$

- $S = GF(2)^3$ length 3 vectors with modulo 2 arithmetic

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

How would you write?

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underline{1}\, v_1 + \underline{1}\, v_2 + \underline{0}\, v_3$$