Norms

Monday, January 23, 2017 9:17 AM

Norms add the idea of length and distance and similarity to a vector space.

Definition: A norm $\|\cdot\|$ on a vector space S is a mapping: $\|\cdot\| : S \to \mathbb{R}$
with the following properties for $\underline{x}, \underline{y} \in S$

1. $\|\underline{x}\| \geq 0$ and $\|\underline{x}\| = 0 \Leftrightarrow \underline{x} = \underline{0}$
2. $\|\underline{x} + \underline{y}\| \leq \|x\| + \|y\|$ (triangle inequality)
3. $\|a \cdot \underline{x}\| = |a| \cdot \|\underline{x}\|$ for any scaler a

absolute value

The length of $\underline{x} \in S$ is $\|\underline{x}\|$

This distance between $\underline{x}$ and $\underline{y}$ is $\|x - y\|$

A linear vector space with an associated norm is called a normed vector space.

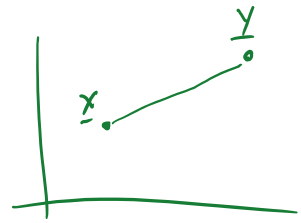
Examplos

1. $S = \mathbb{R}^N$ — l_2 or Euclidean norm

$$\|\ \| = \left(\sum^{N} |\ |^2 \right)^{1/2} \qquad y$$

$$\|\underline{x}\|_2 = \left(\sum_{n=1}^{N} |x_n|^2 \right)^{1/2}$$

corresponds to energy or power (when squared)

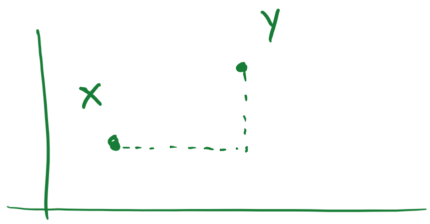


$$\|x-y\|_2 = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

2. $S = \mathbb{R}^N$

$$\|\underline{x}\|_1 = \sum_{n=1}^{N} |x_n|$$

l_1 norm or "taxicab norm" or "Manhattan norm"



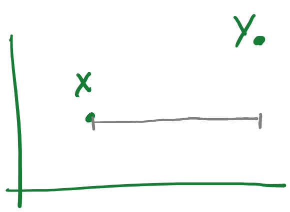
$$\|x-y\|_1 = |x_1 - y_1| + |x_2 - y_2|$$

3. $S = \mathbb{R}^N$

$$\|\underline{x}\|_\infty = \max_{n=1,\dots,N} |x_n|$$

The l_∞ norm or "Chebyshev norm"

$\mathbb{R}^2$



$$\|\underline{x}-\underline{y}\|_\infty = \max\left(|x_1-y_1|, |x_2-y_2|\right)$$

4. $S = \mathbb{R}^N$

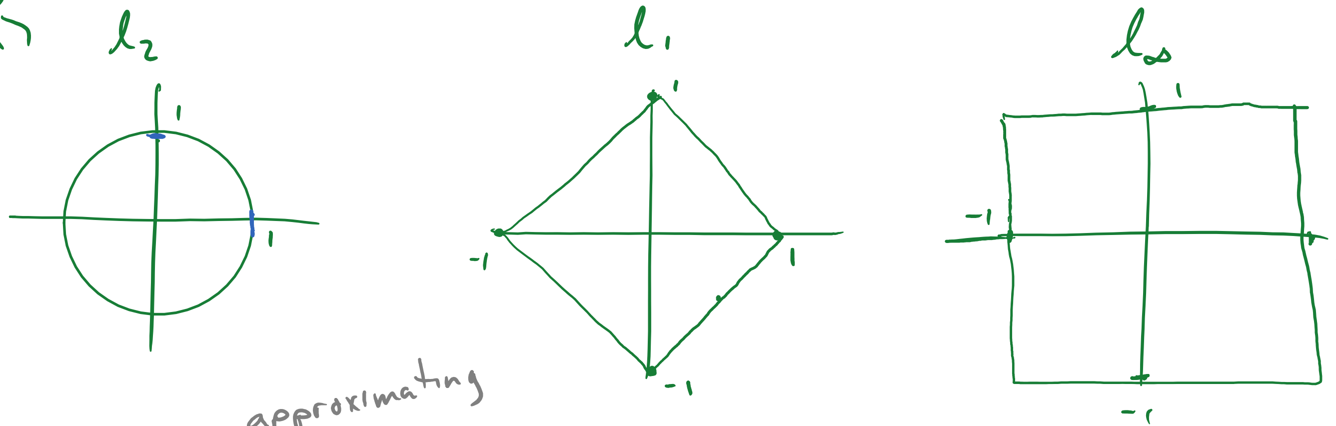
$$\|\underline{x}\|_p = \left(\sum_{n=1}^{N} |x_n|^p \right)^{1/p} \quad \text{for some } 1 \le p < \infty$$

This is the l_p-norm

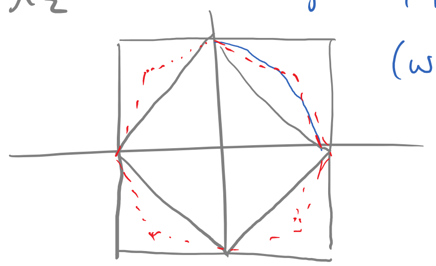
balls in $\mathbb{R}^2$

l_2 l_1 l_∞

unit balls (norm of 1)

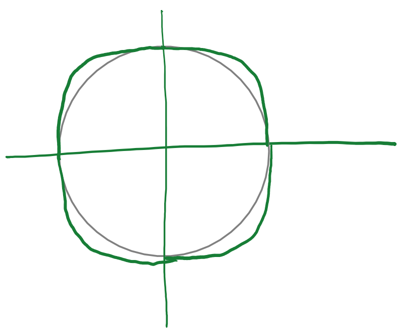


approximating l_2 ≈ average l_1 & l_∞



(weighted average works even better)

l_3 norm



5. S = continuous-time signals on the real line

$$\|x(t)\|_2 = \left(\int_{-\infty}^{\infty} |x(t)|^2 \, dt \right)^{1/2}$$

L_2 norm

$\|x(t)\|_2^2$ is the energy in the signal

6. S = cont.-time functions on $[a,b]$

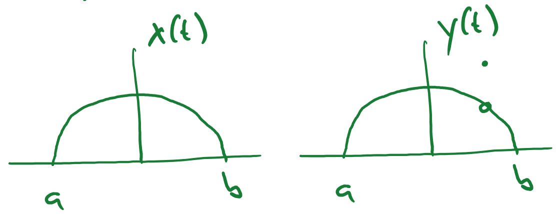
$$\|x(t)\|_p = \left(\int_a^b |x(t)|^p \, dt \right)^{1/p}$$

$$\| x(t) \|_p = \left(\int_a |x(t)| \, dt \right)$$

In a normed linear space, we say that

$$\underline{x} = \underline{y} \quad \text{if} \quad \| \underline{x} - \underline{y} \| = 0$$

for example in $L_2([a,b])$



$y(t)$ differs from $x(t)$ at a single point

then

$$\| \underline{x} - \underline{y} \|_2 = \left(\int_a^b |x(t) - y(t)|^2 \, dt \right)^{1/2} = 0$$

$x(t) = y(t)$ if they differ at only a finite number of points