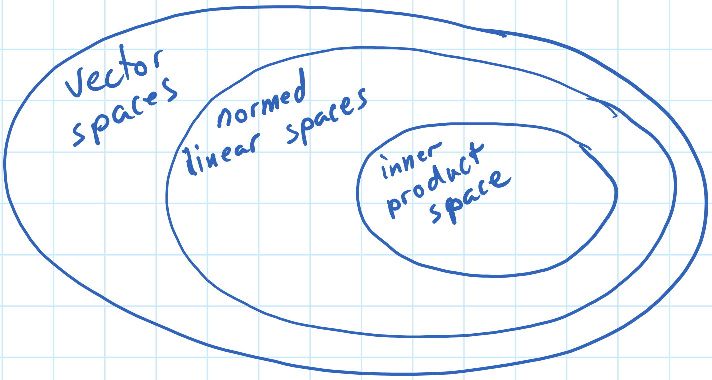
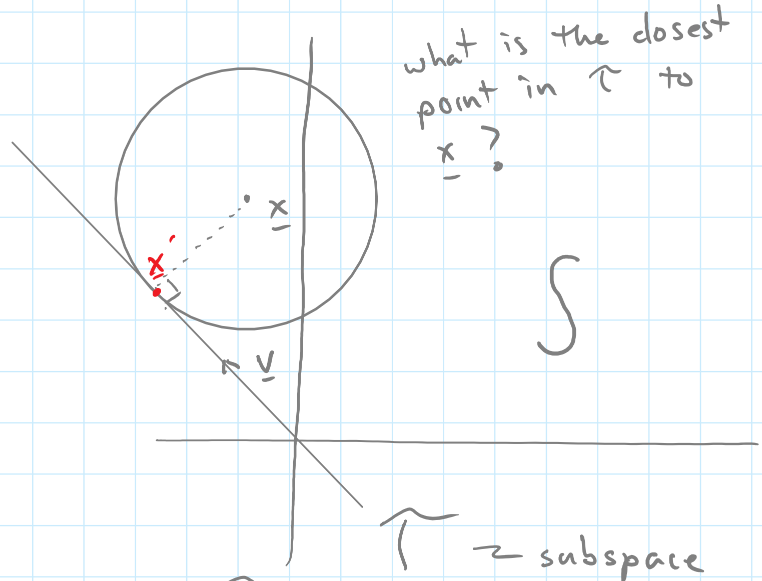
Inner Product Spaces

Monday, January 23, 2017 10:00 AM





note, $\underline{x} - \underline{x}' \perp \underline{v}$ for all $\underline{v} \in T$

$\hookrightarrow$ inner products allow us to formalize this and find $\underline{x}'$ easily.

Definition: An inner product on a (real or complex-valued) vector space S is a mapping

$$\langle \cdot , \cdot \rangle : S \times S \to \mathbb{C}$$

that obeys

complex conjugate of a scalar

1. $\langle \underline{x}, \underline{y} \rangle = \overline{\langle \underline{y}, \underline{x} \rangle}$

2. For any $a, b \in \mathbb{C}$

$$\langle a\underline{x} + b\underline{y}, \underline{z} \rangle = a \langle \underline{x}, \underline{z} \rangle + b \langle \underline{y}, \underline{z} \rangle$$

3. $\langle \underline{x}, \underline{x} \rangle \geq 0$ and $\langle \underline{x}, \underline{x} \rangle = 0 \iff \underline{x} = \underline{0}$

Aside: $\langle \underline{x}, b\underline{y} \rangle = \overline{\langle b\underline{y}, \underline{x} \rangle} = \overline{b \langle \underline{y}, \underline{x} \rangle}$

$= \bar{b} \cdot \overline{\langle y, x \rangle} = \bar{b} \cdot \langle x, y \rangle$

conjugated when we "pull it out"

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standard examples

1) $S = \mathbb{R}^N$ $\quad \langle \underline{x}, \underline{y} \rangle = \sum_{n=1}^{N} x_n y_n = y^T x$

2) $S = \mathbb{C}^N$ $\quad \langle \underline{x}, \underline{y} \rangle = \sum_{n=1}^{N} x_n \bar{y}_n = \underline{y}^H \underline{x}$ — conjugate transpose

3) $S = L_2([a,b])$ $\quad \langle \underline{x}, \underline{y} \rangle = \int_a^b x(t)\, \bar{y}(t)\, dt$

4) $S = \mathbb{R}^{M \times N}$ $\quad \langle X, Y \rangle = \text{trace}(Y^T X) = \sum_{m=1}^{M} \sum_{n=1}^{N} X_{m,n} Y_{m,n}$

trace: sum of entries along the diagonal

called the Frobenius inner product

also called the Hilbert-Schmidt inner product

also called the trace inner product.

$$\left\langle \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} \right\rangle$$

$$\text{trace}\left\{ \begin{pmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \right\} = \boxed{b_{11}a_{11} + b_{21}a_{21} + b_{31}a_{31} + b_{12}a_{12} + b_{22}a_{22} + b_{32}a_{32}}$$

5) S = zero-mean Gaussian random variables with finite variance

$$\langle X, Y \rangle = E\{XY\}$$

6) S = differentiable real-valued continuous-time signals on $\mathbb{R}$

$$\langle \underline{x}, \underline{y} \rangle = \int x(t)y(t)\,dt + \int x'(t)\,y'(t)\,dt$$

where $x'(t) = \frac{d}{dt}x(t)$ called the Sobolev inner product

A linear vector space equipped with an inner product is called an <u>inner product space.</u>

A valid inner product induces a valid norm by

$$\|\underline{x}\| = \sqrt{\langle \underline{x}, \underline{x} \rangle}$$

Properties of induced norms

In addition to the triangle inequality

$$\|\underline{x} + \underline{y}\| \le \|\underline{x}\| + \|\underline{y}\|$$

induce norms have some additional properties

1. Cauchy-Schwarz Inequality

$$|\langle \underline{x}, \underline{y} \rangle| \le \|\underline{x}\| \cdot \|\underline{y}\|$$

$$\frac{|\langle x, y \rangle|}{\|x\| \cdot \|y\|} \le 1$$

$$|\langle \underline{x}, \underline{y} \rangle| = \|\underline{x}\| \cdot \|\underline{y}\| \qquad \|x\| \cdot \|y\|$$

equality is achieved iff $\underline{x}$ and $\underline{y}$ are colinear:

$$\exists\, a \in \mathbb{C} \text{ such that } \underline{y} = a\, \underline{x}$$

2. Pythagorean Theorem

orthogonality or "right angle" condition

$$\langle \underline{x}, \underline{y} \rangle = 0 \;\Rightarrow\; \|\underline{x}+\underline{y}\|^2 = \|\underline{x}\|^2 + \|\underline{y}\|^2$$

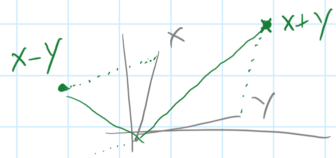
we also have $\|\underline{x}-\underline{y}\|^2 = \|\underline{x}\|^2 + \|\underline{y}\|^2$

because if $\langle x, y \rangle = 0$, then $\langle x, -y \rangle = 0$

3. Parallelogram Law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

prove by starting $\|x+y\|^2 = \langle x+y, x+y \rangle$ & expand ...



4. Polarization Identity

$$\mathrm{Re}\{\langle x, y \rangle\} = \frac{\|x+y\|^2 - \|x-y\|^2}{4}$$

① $\|x+y\|^2 = \langle x+y, x+y \rangle$ $\qquad$ $\mathrm{Re}\{\langle x,y \rangle\} = \frac{1}{2}\left(\langle x,y \rangle + \overline{\langle x,y \rangle}\right)$

$\|x-y\|^2 = \langle x-y, x-y \rangle$

difference

① $\langle x,x \rangle + \langle x,y \rangle + \langle y,x \rangle + \langle y,y \rangle$

$\Big\}\; 2\langle x,y \rangle + 2\langle y,x \rangle$

② $\langle x,x \rangle - \langle x,y \rangle - \langle y,x \rangle + \langle y,y \rangle$

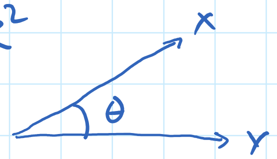
$$② \quad \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle \Big\} \quad 2\langle x,y\rangle + 2\langle y,x\rangle$$

$$= 2\left(\langle x,y\rangle + \overline{\langle x,y\rangle}\right)$$

$$\mathrm{Re}\{\langle x,y\rangle\} = \frac{1}{4}\left(\|x+y\|^2 + \|x-y\|^2\right)$$

Angles between vectors

in $\mathbb{R}^2$



$$\langle x,y\rangle = \|x\|\cdot\|y\|\cos\theta$$

We can extend this idea to any inner product space

Definition: The angle between two vectors $\underline{x}$ and $\underline{y}$ in an inner product space by

$$\cos\theta = \frac{\langle \underline{x},\underline{y}\rangle}{\|\underline{x}\|\cdot\|\underline{y}\|}$$

Definition: Vectors $\underline{x}$ and $\underline{y}$ in an inner product space are orthogonal to one another if

$$\langle \underline{x},\underline{y}\rangle = 0$$

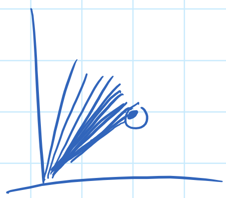
Aside: Hilbert Spaces

An inner product space is a Hilbert space if it is complete. That is, for every sequence

$\underline{x}_1, \underline{x}_2, \ldots \in S$ for which

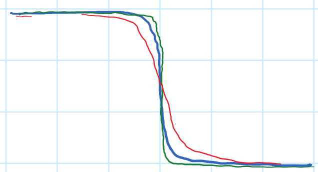
$$\lim_{\min(m,n)\to\infty} \|\underline{x}_m - \underline{x}_n\| = 0 \quad \text{will also have}$$

$$\lim_{n\to\infty} \underline{x}_n = \underline{x}^* \in S \quad \text{where } \|\cdot\| \text{ is the induced norm.}$$



$\mathbb{R}^N$ with $\langle \cdot, \cdot \rangle$

What about the set of all continuous, bounded functions on $[0, 1]$?



no, not a Hilbert space.

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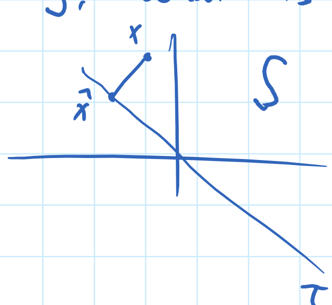
Let S be a Hilbert space, and T be a subspace of S. Given a $\underline{x} \in S$, what is the closest point $\hat{\underline{x}} \in T$?



In other words, find $\hat{x} \in T$ that minimizes $\|\underline{x} - \hat{\underline{x}}\|$.

$$\min_{y \in T} \|x - y\| \qquad (1)$$

norm induced by inner product

$\Rightarrow$ This has a unique solution characterized by the orthogonality principle

Theorem: Let $\mathcal{S}$ be a Hilbert space, and $\mathcal{T}$ be a finite-dimensional subspace. Given an arbitrary $\underline{x} \in \mathcal{S}$

actually works for $\mathcal{T}$ infinite-dimensional and closed

1. there is exactly one $\hat{\underline{x}} \in \mathcal{T}$ such that

$$\underline{x} - \hat{\underline{x}} \perp \mathcal{T},$$

meaning $\langle \underline{x} - \hat{\underline{x}}, \underline{y} \rangle = 0$ for all $\underline{y} \in \mathcal{T}$

2. this $\hat{\underline{x}}$ is the closest point in $\mathcal{T}$ to $\underline{x}$; That is $\hat{\underline{x}}$ is the unique minimizer to equation (1) above.

Proof:

starting with part 2.

let $\hat{\underline{e}} = \underline{x} - \hat{\underline{x}} \perp \mathcal{T}$

let $\underline{y} \in \mathcal{T}$, $\underline{y} \neq \hat{\underline{x}}$; and set $\underline{e} = \underline{x} - \underline{y}$

we will show that

$$\|e\| > \|\hat{e}\| \qquad \left(\|x-y\| > \|x - \hat{x}\| \right)$$

$$\|e\|^2 = \|x-y\|^2 = \|\hat{e} - (y - \hat{x})\|^2$$

$$= \langle \hat{e} - (y - \hat{x}), \hat{e} - (y - \hat{x}) \rangle$$

$$= \|\hat{e}\|^2 + \|y - \hat{x}\|^2 - \langle \hat{e}, (y - \hat{x}) \rangle - \langle (y - \hat{x}), \hat{e} \rangle$$

since $y - \hat{x} \in T$ and $\hat{e} \perp T$

$$\langle \hat{e}, (y - \hat{x}) \rangle = \langle (y - \hat{x}), \hat{e} \rangle = 0$$

$$\|e\|^2 = \|\hat{e}\|^2 + \|y - \hat{x}\|^2$$

but since $y \neq \hat{x}$, this must be > 0

$$\|e\|^2 > \|\hat{e}\|^2$$ QED (for part 2.)

How to compute the best approximation

Let N be the dimension of T, and let $\underline{v}_1, \dots, \underline{v}_N$ be a basis for T. We want to find coefficients $a_1, \dots, a_N \in \mathbb{C}$ such that

$$\underline{\hat{x}} = a_1 \underline{v}_1 + a_2 \underline{v}_2 + \dots + a_N \underline{v}_N$$

The orthogonality principle tells us that

$$\langle x - \hat{x}, \underline{v}_n \rangle = 0 \quad \text{for } n = 1, \dots, N$$

so

$$\langle \underline{x} - \sum_{k=1}^{N} a_k \underline{v}_k, \underline{v}_n \rangle = 0 \quad \text{for } n = 1, \dots, N$$

$$\langle \underline{x}, \underline{v}_n \rangle = \sum_{k=1}^{N} a_k \langle \underline{v}_k, \underline{v}_n \rangle \quad \text{for } n = 1, \dots, N$$

this yields

$$\begin{bmatrix} \langle \underline{v}_1, \underline{v}_1 \rangle & \langle v_2, v_1 \rangle & \cdots & \langle v_N, v_1 \rangle \\ \langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle & \cdots & \langle v_N, v_2 \rangle \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \begin{bmatrix} \langle x, v_1 \rangle \\ \vdots \end{bmatrix}$$

$$\boxed{\underbrace{\begin{bmatrix} \langle \underline{v}_1, \underline{v}_1 \rangle & \langle \underline{v}_2, \underline{v}_1 \rangle & \cdots & \langle \underline{v}_N, \underline{v}_1 \rangle \\ \langle \underline{v}_1, \underline{v}_2 \rangle & \langle \underline{v}_2, \underline{v}_2 \rangle & \cdots & \langle \underline{v}_N, \underline{v}_2 \rangle \\ \vdots & & \ddots & \\ \langle \underline{v}_1, \underline{v}_N \rangle & & \cdots & \langle \underline{v}_N, \underline{v}_N \rangle \end{bmatrix}}_{\substack{\text{Gram matrix} \\ \text{-or-} \\ \text{Grammian of the basis } \{\underline{v}_n\}}} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \langle \underline{x}, \underline{v}_1 \rangle \\ \langle \underline{x}, \underline{v}_2 \rangle \\ \vdots \\ \langle \underline{x}, \underline{v}_N \rangle \end{bmatrix}}$$

G

$$G\underline{a} = \underline{b} \qquad \text{where} \qquad b_n = \langle \underline{x}, \underline{v}_n \rangle$$

$$G_{kn} = \langle \underline{v}_n, \underline{v}_k \rangle$$

1. G is guaranteed to be invertible because $\{\underline{v}_n\}$ are linearly independent

$$\boxed{\underline{a} = G^{-1}\underline{b}}$$

2. G is conjugate symmetric ("Hermitian")

$$G = G^H$$

Proof of Cauchy-Schwarz

set $\quad z = x - \dfrac{\langle x,y\rangle}{\|y\|^2}\, y$

notice that $\quad \langle z, y\rangle = 0 \qquad \left\{ \; \langle z,y\rangle = \langle x,y\rangle - \dfrac{\langle x,y\rangle}{\|y\|^2}\|y\|^2 \right.$

Then $\quad x = z + \dfrac{\langle x,y\rangle}{\|y\|^2}\, y$

and since $\quad y \perp z$

$$\|x\|^2 = \frac{|\langle x,y\rangle|^2}{\|y\|^4}\|y\|^2 + \|z\|^2$$

$$= \frac{|\langle x,y\rangle|^2}{\|y\|^2} + \|z\|^2$$

Thus

$$|\langle x,y\rangle|^2 = \|x\|^2\|y\|^2 - \underline{\|z\|^2}\,\underline{\|y\|^2} \le \|x\|^2\|y\|^2$$

We have equality iff $z = \underline{0}$. If $z = 0$ then

$$x = \alpha y \qquad \left(\alpha = \frac{\langle x,y\rangle}{\|y\|^2}\right)$$

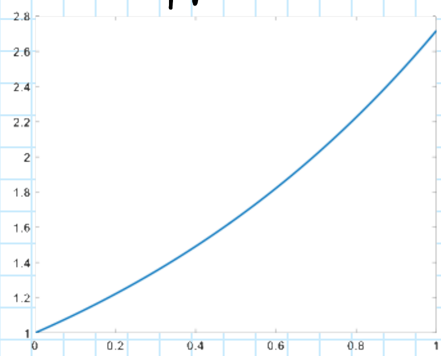
conversely, if $x = \alpha y$ for some $\alpha \in \mathbb{C}$ then

$$z = \alpha y - \frac{\alpha\langle y,y\rangle}{\|y\|^2}\, y = 0$$

Example - Polynomial Approximation of e^t

Wednesday, February 1, 2017 9:00 AM

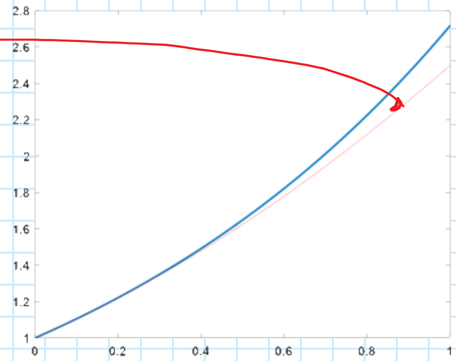
We want to calculate a quadratic approximation of $x(t) = e^t$ over $[0,1]$



we could truncate the Taylor series expansion

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \ldots$$

$$\tilde{x}_{taylor}(t) = 1 + t + \tfrac{1}{2}t^2$$



Try $\tilde{x}(t) = a_1 + a_2 t + a_3 t^2$

minimize

$$\|e^t - \tilde{x}(t)\|_{L_2([0,1])} = \sqrt{\int_0^1 |e^t - \tilde{x}(t)|^2 \, dt}$$

set this up as a subspace approximation

$$v_1(t) = 1 \qquad v_2(t) = t \qquad v_3(t) = t^2$$

set $T = \text{span}\{v_1, v_2, v_3\}$

we need to find the Gram matrix $\left(G_{ij} = \langle v_i, v_j \rangle\right)$

$$\langle v_1, v_1 \rangle = \int_0^1 v_1(t)\bar{v}_1(t)\,dt = \int_0^1 1\,dt = 1$$

$$\langle v_1, v_2 \rangle = \int_0^1 1 \cdot t\,dt = \tfrac{1}{2}$$

$$\langle v_1, v_3 \rangle = \int_0^1 1 \cdot t^2\,dt = \tfrac{1}{3}$$

$$\langle v_2, v_2 \rangle = \int_0^1 t \cdot t\,dt = \tfrac{1}{3}$$

$$\langle v_2, v_3 \rangle = \int_0^1 t \cdot t^2\,dt = \tfrac{1}{4}$$

$$\langle v_3, v_3 \rangle = \int_0^1 t^2 \cdot t^2 = \tfrac{1}{5}$$

$$G = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

$$\underline{a} = G^{-1}\underline{b}$$

$\underline{b} = ?$

Example continued...

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$$b_1 = \langle e^t, v_1 \rangle = \int_0^1 e^t = e - 1$$

$$b_2 = \langle e^t, v_2 \rangle = \int_0^1 t e^t = 1$$

$$b_3 = \langle e^t, v_3 \rangle = \int_0^1 t^2 e^t = e - 2$$

$$\int u\,dv = uv - \int v\,du$$

$u = t \quad dv = e^t dt$

$du = dt \quad v = e^t$

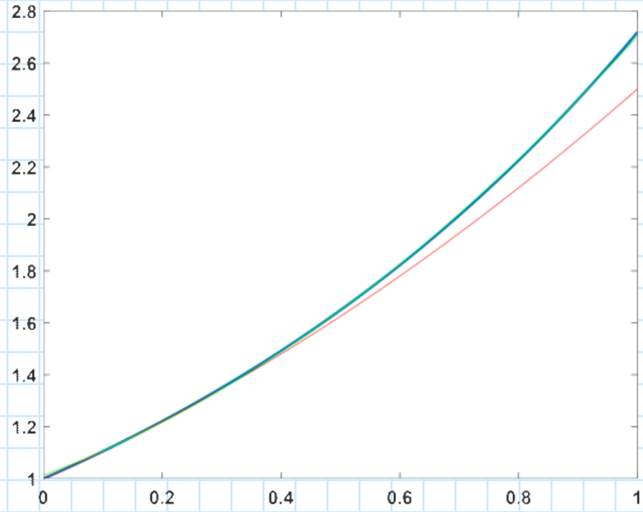
$$t e^t - e^t \Big|_0^1 =$$

$u = t^2 \quad dv = e^t dt$

$du = 2t\,dt \quad v = e^t$

$$t^2 e^t \Big|_0^1 - \underbrace{\int_0^1 2 t e^t dt}_{2}$$

$$a = \begin{bmatrix} 1.01 \\ 0.85 \\ 0.84 \end{bmatrix}$$




```
t=[0:0.01:1];
x=exp(t);
plot(t,x)
xtaylor=1+t+0.5*t.^2;
hold on
plot(t,xtaylor,'r');
G=[1 1/2 1/3; 1/2 1/3 1/4; 1/3 1/4 1/5]
b=[exp(1)-1; 1; exp(1)-2]
a=inv(G)*b
xtilde=polyval(flipud(a),t);
plot(t,xtilde,'g')
```

We could also define another inner product to change our approximation

$$\langle x, y \rangle_s = \int_0^1 w(t)\, x(t)\, y(t)\, dt$$

where $w(t) = 4\left|t - \tfrac{1}{2}\right|$ $\leftarrow$ minimizes error at ends