Orthogonal Bases

Monday, January 30, 2017 9:01 AM

A collection of vectors $\{\underline{v}_1, \ldots, \underline{v}_N\}$ in a finite-dimensional vector space S is called an **orthogonal basis** if

1. $\text{span}(\{\underline{v}_1, \ldots, \underline{v}_N\}) = S$

2. $\underline{v}_j \perp \underline{v}_k$ (i.e. $\langle \underline{v}_j, \underline{v}_k \rangle = 0$) for all $j \neq k \in 1, \ldots, N$

If $\|\underline{v}_n\| = 1$ for $n = 1, \ldots, N$ then we call it an **orthonormal basis** or **orthobasis**

A note on infinite dimensions

if $\mathbb{B} = \{\underline{v}_n\}_{n \in \mathbb{Z}}$ is an infinite sequence of orthonormal vectors, it is an orthobasis of the closure of $\text{span}(\mathbb{B})$.

example: Let $x(t)$ be any function on $[0,1]$ that is not a polynomial, e.g. $x(t) = \sin(2\pi t)$

Let $\mathbb{B} = \{1, t, t^2, \ldots\}$; the span $\mathbb{B}$ is all polynomials on $[0,1]$. so $\underline{x} \notin \text{span}(\mathbb{B})$. But we can approximate $x(t)$ arbitrarily well by elements in $\mathbb{B}$ so $x(t) \in \text{closure}(\text{span}(\mathbb{B}))$

Examples of orthobases

$S = \mathbb{R}^2$ $\qquad \underline{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \underline{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Orthobases 2

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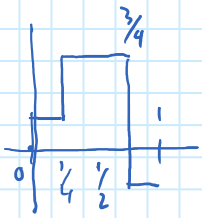
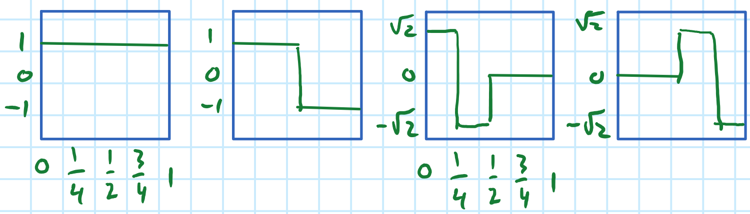
examples cont.

2. S = space of piecewise constant functions on $[0,\tfrac14), [\tfrac14,\tfrac12), [\tfrac12,\tfrac34), [\tfrac34,1]$

e.g.



The following four functions form an orthobasis



3. Sampling

$\mathbb{B}_{\pi/T}(\mathbb{R})$ = real-valued functions which are band-limited to π/T, equipped with the standard inner product. The set of functions

$$\left\{ \underline{v}_n = \sqrt{T}\,\frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)},\ n\in\mathbb{Z}\right\}$$

is an orthobasis for $\mathbb{B}_{\pi/T}(\mathbb{R})$

$$\left\langle \sqrt{T}\,\frac{\sin(\pi(t-n_1T)/T)}{\pi(t-n_1T)},\ \sqrt{T}\,\frac{\sin(\pi(t-n_2T)/T)}{\pi(t-n_2T)}\right\rangle$$

$$= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} T e^{-j\Omega T n_1} e^{j\Omega T n_2}\, d\Omega \qquad \text{(Parseval)}$$

$$= \begin{cases} 1 & n_1 = n_2 \\ 0 & n_1 \neq n_2 \end{cases}$$

Linear approximation and orthobases

Wednesday, February 1, 2017 9:37 AM

Best linear approximation problem again:

Given $x \in S$, we want to find the closest point in a subspace T. If T has an orthobasis $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_N\}$ then $\hat{x} = a_1 \underline{v}_1 + a_2 \underline{v}_2 + \cdots + a_N \underline{v}_N$

$$\underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}}_{\underline{a}} = G^{-1} \underline{b} \qquad \text{but if} \quad \langle v_i, v_j \rangle = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}$$

Then $G = I$

$$\text{so} \quad \underline{a} = \underline{b} = \begin{bmatrix} \langle \underline{x}, \underline{v}_1 \rangle \\ \langle \underline{x}, \underline{v}_2 \rangle \\ \vdots \end{bmatrix}$$

$$\boxed{\hat{\underline{x}} = \sum_{n=1}^{N} \langle \underline{x}, \underline{v}_n \rangle \underline{v}_n}$$

This also works for infinite-dimensional subspaces

$$\underline{x} = \sum_{n=1}^{\infty} \langle \underline{x}, \underline{v}_n \rangle \underline{v}_n$$

Generalizing — any point $\underline{x} \in S$ can be "approximated" perfectly as

$$\boxed{\underline{x} = \sum_{n} \langle \underline{x}, \underline{v}_n \rangle \underline{v}_n}$$

transform coefficients.

We can recreate a vector in a Hilbert space from a sequence of numbers $\{\langle \underline{x}, \underline{v}_n \rangle\}$.

We can think of every different orthobasis of S as a transform and $\{\langle \underline{x}, \underline{v}_n \rangle\}$ are the transform coefficients

Parseval's Theorem

Wednesday, February 1, 2017 9:49 AM

With respect to the Fourier transform:

$$\|x(t)\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 \, d\Omega = \frac{1}{2\pi} \|X(j\Omega)\|_2^2$$

There is something similar for any orthobasis expansion.

Let S be a Hilbert space with $\langle \cdot, \cdot \rangle_S$ which induces a norm $\|\cdot\|_S$. Let $\{\underline{v}_k\}_{k \in \Gamma}$ be an orthobasis for S. Then for every $\underline{x}, \underline{y} \in S$,

$$\boxed{\langle \underline{x}, \underline{y} \rangle_S = \sum_{k \in \Gamma} \alpha_k \overline{\beta}_k}$$

where $\alpha_k = \langle \underline{x}, \underline{v}_k \rangle_S$ and $\beta_k = \langle \underline{y}, \underline{v}_k \rangle_S$

You can think of the α's and β's as transform coefficients of $\underline{x}$ & $\underline{y}$ respectively.

so

$$\langle \underline{x}, \underline{y} \rangle_S = \langle \underline{\alpha}, \underline{\beta} \rangle_{\ell_2}$$

$$\|\underline{x}\|_S^2 = \|\underline{\alpha}\|_2^2$$

notice, no $\frac{1}{2\pi}$ factor

Thus, every Hilbert space with an orthobasis is equivalent to ℓ_2

— lengths, angles, relations all map to ℓ_2

Proof of Parseval's Theorem

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Let $\alpha_k = \langle \underline{x}, \underline{v}_k \rangle$ and $\beta_k = \langle \underline{y}, \underline{v}_k \rangle$

write

$$\underline{x} = \sum_{k\in\Gamma} \alpha_k \underline{v}_k \quad \text{and} \quad \underline{y} = \sum_{k\in\Gamma} \beta_k \underline{v}_k$$

so

$$\langle \underline{x}, \underline{y} \rangle_S = \left\langle \sum_{k\in\Gamma} \alpha_k \underline{v}_k , \sum_{\ell\in\Gamma} \beta_\ell \underline{v}_\ell \right\rangle_S$$

$$= \sum_{k\in\Gamma} \alpha_k \left\langle \underline{v}_k , \sum_{\ell\in\Gamma} \beta_\ell \underline{v}_\ell \right\rangle_S$$

$$= \sum_{k\in\Gamma} \sum_{\ell\in\Gamma} \alpha_k \overline{\beta_\ell} \langle \underline{v}_k , \underline{v}_\ell \rangle_S$$

for any fixed k, only one term of the inner sum $\neq 0$

$$\boxed{\langle \underline{x}, \underline{y} \rangle_S = \sum_{k\in\Gamma} \alpha_k \overline{\beta_k}}$$

Distances in Hilbert Spaces

Wednesday, February 1, 2017 10:14 AM

$$\|\underline{x}-\underline{y}\|_S = \|\underline{\alpha}-\underline{\beta}\|_2 = \left(\sum_k (\alpha_k-\beta_k)^2\right)^{1/2}$$

so a small change in a vector in S will yield a small change in the orthobasis expansion coefficients (a.k.a. transform coefficients)

suppose $\{\alpha_k = \langle \underline{x}, \underline{v}_k \rangle_S\}$ are the transform coefficients of $\underline{x}$

let $\tilde{\alpha}_k = \begin{cases} \alpha_{k_0}+\delta & k=k_0 \\ \alpha_k & k \neq k_0 \end{cases}$ ← localized error

then $\tilde{x} = \sum_k \tilde{\alpha}_k \underline{v}_k$

diffused error

but it's the same

$$\|x-\tilde{x}\|_S = \|\underline{\alpha}-\underline{\tilde{\alpha}}\|_2 = \delta \qquad \text{so} \quad \|x-\tilde{x}\|_S = \delta$$

This is the distance across the whole signal

Example

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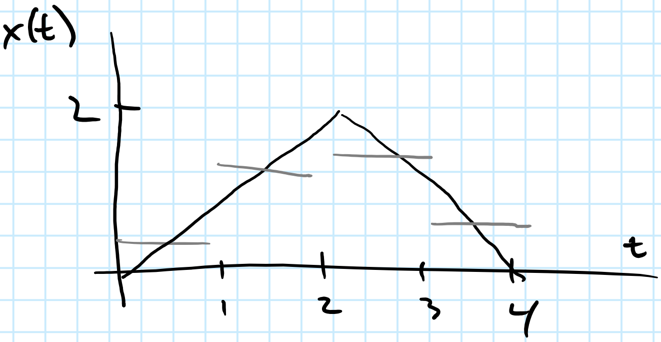


$x(t) \in \underline{L_2([0,4])}$

$$x(t) = \begin{cases} t & 0 \le t \le 2 \\ 4-t & 2 \le t \le 4 \end{cases}$$

$\mathcal{T}$ = piecewise constant functions on $[0,1), [1,2), [2,3), [3,4]$

$$V_n(t) = \begin{cases} 1 & (n-1) \le t < n \\ 0 & \text{otherwise} \end{cases}$$

$n = 1,2,3,4$

$$\sum_n a_n V_n(t)$$

$$\boxed{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \underbrace{G^{-1}}_{\text{Gram mtx}} \cancel{b}} \quad \begin{bmatrix} \langle x, v_1 \rangle \\ \langle x, v_2 \rangle \\ \vdots \end{bmatrix}$$

This will be the identity matrix

$$\langle V_k, V_j \rangle = \int_0^4 V_k V_j(t)\, dt$$

$$\langle V_1, V_1 \rangle = \int_0^1 1 \cdot 1\, dt = 1 \qquad \langle V_1, V_2 \rangle = \int_0^1 1 \cdot 0\, dt + \int_1^2 0 \cdot 1\, dt = 0$$

... orthobasis

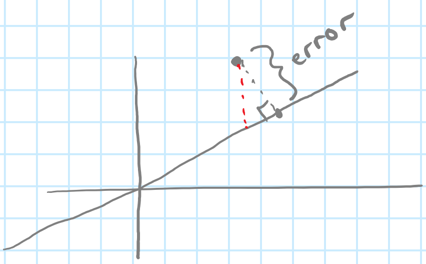
$$a_i = \langle x, V_i \rangle$$

$$a_1 = \langle x, V_1 \rangle = \int_0^1 t\, dt = \frac{1}{2}$$

$$a_2 = \langle x, V_2 \rangle = \int_1^2 t\, dt = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

$$a_3 = \langle x, V_3 \rangle = \int_2^3 (4-t)\, dt = 4 - \left(\frac{9}{2} - \frac{4}{2}\right) = \frac{3}{2}$$

$$a_4 = \langle x, V_4 \rangle = \int_3^4 (4-t)\, dt = 1/2$$



Lengths

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$\{V_k\}$ is an orthobasis on an inner product space

$$x = \sum_k \langle x, V_k \rangle V_k$$

$$\|x\|^2 = \langle x, x \rangle$$

$$= \left\langle \sum_k \langle x, V_k \rangle V_k \, , \, \sum_\ell \langle x, V_\ell \rangle V_\ell \right\rangle$$

$$= \sum_k \langle x, V_k \rangle \left\langle V_k \, , \, \sum_\ell \langle x, V_\ell \rangle V_\ell \right\rangle$$

$$= \sum_k \langle x, V_k \rangle \sum_\ell \overline{\langle x, V_\ell \rangle} \underbrace{\langle V_k , V_\ell \rangle}_{= \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{else} \end{cases}}$$

$$= \sum_k \langle x, V_k \rangle \overline{\langle x, V_k \rangle}$$

$$= \sum |\langle x, V_k \rangle|^2$$

Parseval again

Truncating Ortho Expansions

Monday, February 6, 2017 8:40 AM

Say $\{v_k\}_{k=0}^{\infty}$ is an orthobasis for a Hilbert space S.

T is a subspace spanned by the first K elements of $\{v_k\}$

$$T = \text{span}\left(\{v_0, v_1, \ldots, v_{K-1}\}\right)$$

1) what is the closest point in T (call it $\hat{x}$) to a point $x \in S$?

$$\hat{x} = \sum_{k=0}^{K-1} \langle x, v_k \rangle v_k$$

we've already shown this

2) how good is the approximation of $\hat{x}$ to x?

$$\|x - \hat{x}\|_S^2 = \left\| \sum_{k=0}^{\infty} \langle x, v_k \rangle v_k - \sum_{k=0}^{K-1} \langle x, v_k \rangle v_k \right\|_S^2$$

$$= \left\| \sum_{k=K}^{\infty} \langle x, v_k \rangle v_k \right\|^2$$

$$= \sum_{k=K}^{\infty} |\langle x, v_k \rangle|^2$$

if the first K coefficients are much larger than those from K to ∞, then $\hat{x}$ is a good approximation of x and we can "compress" x using the first K terms.

The main idea in video and image compression --- coming up.

Gram-Schmidt

Monday, February 6, 2017 9:39 AM

The goal is to take a sequence of signals $\{v_1, \ldots, v_N\}$ and produce $\{u_1, \ldots, u_N\}$ such that

$$\text{span}(\{v_1, \ldots, v_N\}) = \text{span}(\{u_1, \ldots, u_N\})$$

and

$$\langle u_n, u_m \rangle = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Assume $\{v_n\}$ forms a basis for an N-dimensional space or subspace.

1) choose $w_1 = v_1$ and normalize it to get

$$u_1 = \frac{w_1}{\|w_1\|}$$

2) To get u_2, we subtract from v_2 its projection onto u_1

$$w_2 = v_2 - \langle v_2, u_1 \rangle u_1$$

$$u_2 = \frac{w_2}{\|w_2\|}$$

$$\langle u_2, u_1 \rangle = \frac{1}{\|w_2\|} \langle w_2, u_1 \rangle$$

$$= \frac{1}{\|w_2\|} \left(\langle v_2, u_1 \rangle - \langle v_2 u_1 \rangle \overbrace{\cancel{\langle u_1, u_1 \rangle}}^{1} \right)$$

$$= 0$$

So $\{u_1, u_2\}$ is an orthobasis for $\text{span}(\{v_1, v_2\})$

Gram-Schmidt II

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3) At the beginning of the k^{th} step, $\{u_1, \dots, u_{k-1}\}$ is an orthobasis for span$(\{v_1, \dots, v_{k-1}\})$.

So we u_k by subtracting the projection of v_k onto the span$(\{u_1, \dots, u_{k-1}\})$ and normalizing:

$$w_k = v_k - \sum_{l=1}^{k-1} \langle v_k, u_l \rangle u_l$$

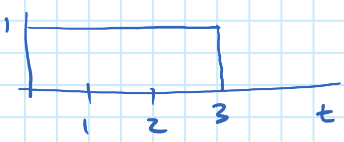
$$u_k = \frac{w_k}{\|w_k\|}$$

By induction, $\{u_1, \dots, u_k\}$ is and orthobasis for span$(\{v_1, \dots, v_k\})$

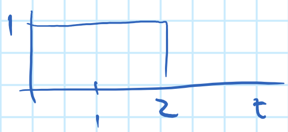
If, at any point, $w_k = 0$, then the set $\{v_k\}$ was not a basis — just toss that vector and continue.

Example: Let S be the set of piecewise-constant signals on $[0,1), [1,2), [2,3]$ with the standard inner product, L_2

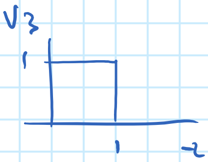
$\underline{v}_1 =$



v_2



v_3

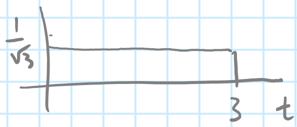


$w_1 = v_1$

$$\|w_1\| = \sqrt{\langle w_1, w_1 \rangle} = \left(\int_0^3 1 \cdot 1 \, dt \right) = \sqrt{3}$$

$$u_1 = \frac{w_1}{\|w_1\|}$$

$u_1 =$



Gram-Schmidt example continued

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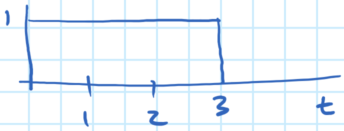
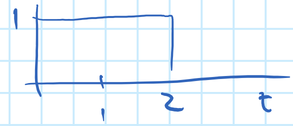
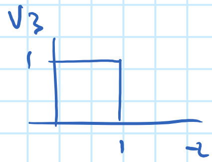
$v_1 =$



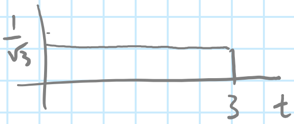
v_2



v_3



$u_1 =$

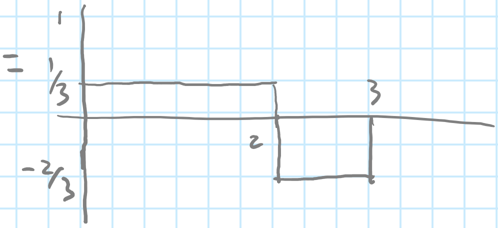


$$\langle v_2, u_1 \rangle = \int_0^2 \frac{1}{\sqrt{3}}\,dt = \frac{2}{\sqrt{3}}$$

$$w_2 = v_2 - \langle v_2, u_1 \rangle u_1$$

$$\langle v_2, u_1 \rangle u_1 = \frac{2}{3} \quad \text{for } t = 0 \text{ to } 3$$

$w_2 =$



$$\|w_2\|^2 = \frac{2}{9} + \frac{4}{9} = \frac{6}{9} = \frac{2}{3}$$

$$\|w_2\| = \sqrt{2/3}$$

$$u_2 = \begin{cases} \frac{1}{\sqrt{6}} & 0 \le t < 2 \\ -\frac{2}{\sqrt{6}} & 2 \le t < 3 \end{cases}$$

$$\langle v_3, u_1 \rangle = 1/\sqrt{3}$$

$$\langle v_3, u_2 \rangle = 1/3$$

$$w_3 = v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2$$

$$= \begin{cases} 1 - 1/3 - 1/9 = \frac{5}{9} & 0 \le t < 1 \\ -1/3 - 1/9 = -4/9 & 1 \le t < 2 \\ -1/3 + 2/9 = -1/9 & 2 \le t \le 3 \end{cases}$$

$$u_3 = \frac{w_3}{\|w_3\|} \qquad \|w_3\|^2 = 25/81 + 16/81 + 1/81 = 42/81 = 14/27$$

$$u_3 = \sqrt{\frac{27}{14}} \cdot w_3 = \frac{3\sqrt{3}}{\sqrt{14}} w_3 = \begin{cases} 5/\sqrt{42} & 0 \le t < 1 \\ -4/\sqrt{42} & 1 \le t < 2 \\ -1/\sqrt{42} & 2 \le t \le 3 \end{cases}$$

Gram-Schmidt example 2

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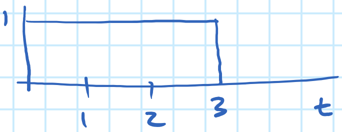
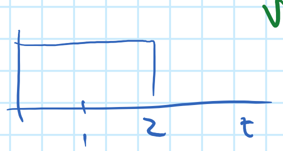
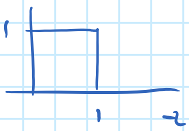
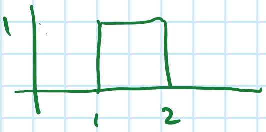
$V_3 =$ 

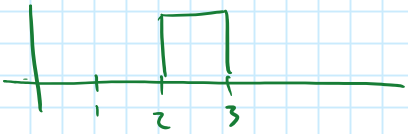
V_2 

V_1 

Start at the other end...

$u_1 = V_1$

$w_2 =$  $\Rightarrow u_2 = w_2$

$w_3 =$  $\Rightarrow u_3 = w_3$