Cosine Transform IV

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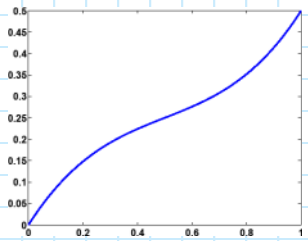
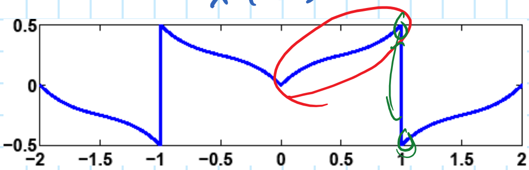
Cosine-IV is different from, but related to the cosine-I transform that we have used

$x(t)$



$\tilde{x}(t)$



$$\tilde{x}(t) = \begin{cases} -x(t+2) & -2 \le t < 1 \\ x(-t) & -1 \le t < 0 \\ x(t) & 0 \le t \le 1 \\ -x(2-t) & 1 < t \le 2 \end{cases}$$

$\tilde{x}(t)$ has even symmetry around $t=0$ and odd symmetry around $t=-1$ and $t=1$

using Fourier series

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{\pi k t}{2}\right) + \cancel{\sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi k t}{2}\right)}$$

even symmetry about $t=0$

every other term disappears because of odd symmetry about $t=1,-1$

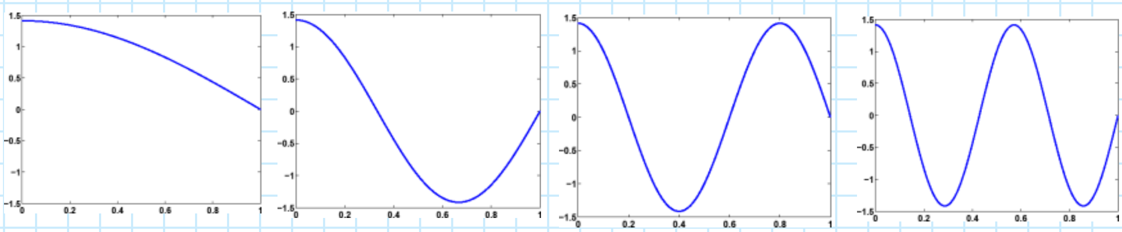
$a_k = 0$ for k even

Cosine-IV basis

$$\psi_k(t) = \sqrt{2}\cos\left(\left(k+\tfrac{1}{2}\right)\pi t\right) \quad \text{for } k = 0, 1, 2, \ldots$$

make $\psi_k(t)$ unit norm

orthonormal



Discrete Cosine IV transform & Block Transforms

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$$\psi_k[n] = \sqrt{\frac{2}{N}} \cos\left(\frac{\pi}{N}\left(k+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\right), \quad k=0,1,\dots,N-1$$

↑ can be computed using a fast algorithm that uses the FFT

In general, we want to represent signals on arbitrarily long time intervals.

↳ break the real line into intervals and build up each interval separately. (similar to blocking in JPEG)

Suppose $\{\psi_k(t),\ k \geq 0\}$ is an orthobasis for $L_2([0,1])$. → Then for a fixed integer, n,

$\{\psi_k(t-n),\ k \geq 0\}$ is an orthobasis for $L_2([n,n+1])$

Define $\psi_{n,k}(t) = \psi_k(t-n)$ then

$\{\psi_{n,k}(t),\ n \in \mathbb{Z},\ k \geq 0\}$ is an orthobasis for $L_2(\mathbb{R})$

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \langle x(t), \psi_{n,k}(t) \rangle \, \psi_{n,k}(t)}$$

Block transform with coefficients

$$\alpha_{n,k} = \langle x(t), \psi_{n,k}(t) \rangle$$

"time index", "horizontal index"

"frequency index" or "vertical index"

DCT expansion coefficients

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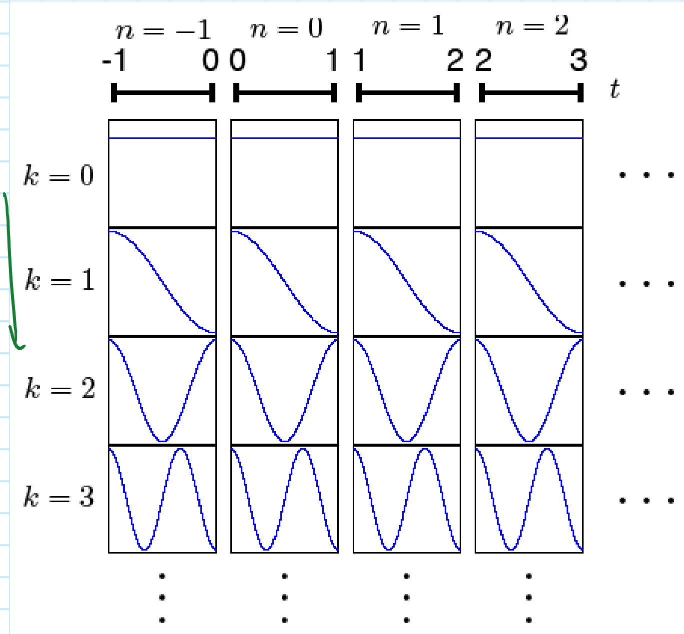
time →

freq. coef.



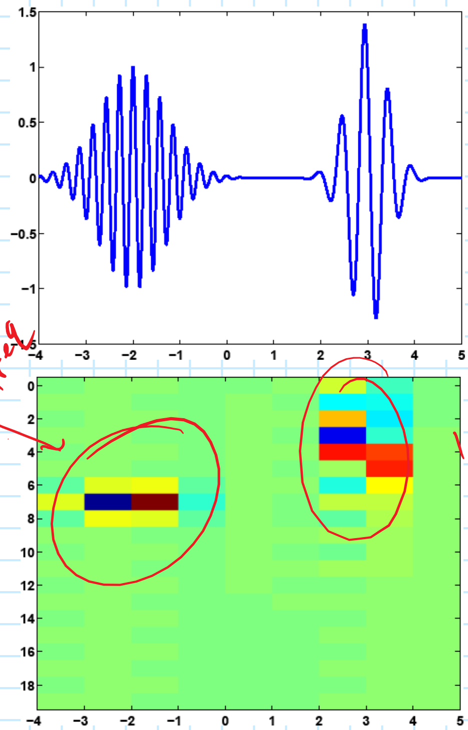
Example

$x(t)$

higher freq

lower freq.

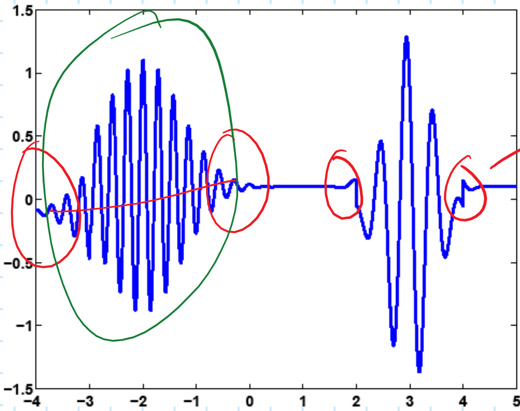
k ↓



$\{\alpha_{n,k}\}$ $n \rightarrow$

What happens if we perturb the coefficients slightly? if we add ± 0.1 to the $k=0$ coefficients

sharp discontinuities



Discontinuities are perceptually very bad

what to do?

Lapped Orthogonal Transform (LOT)

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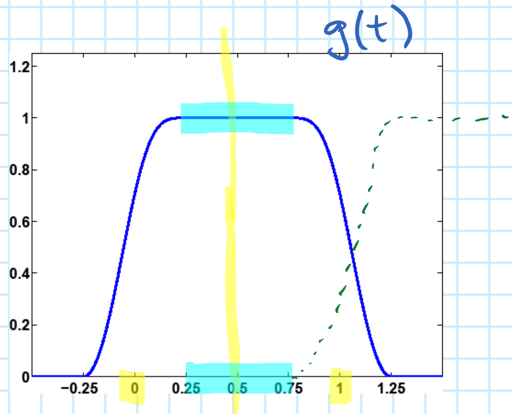
(also called the modified discrete cosine transform)

The LOT is a modified cosine-IV transform that breaks the signal into overlapping frames.

It essentially works by multiplying the signal by a smooth window function, and take the cosine transform of the result. The trick is maintaining orthogonality.

window function

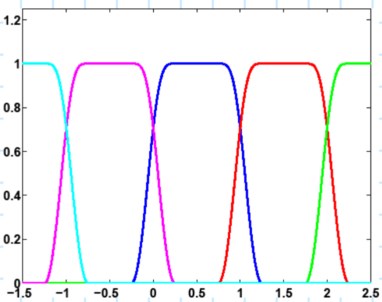
$g(t)$



1. $g(t) = 0$ for $t < -\frac{1}{4}$ & $t > \frac{5}{4}$
2. $g(t) = 1$ for $\frac{1}{4} \le t \le \frac{3}{4}$
3. $g(t)$ is symmetric around $t = \frac{1}{2}$
4. $g(t)$ is monotonically increasing $-\frac{1}{4} \le t \le \frac{1}{4}$
5. $|g(t)|^2 + |g(t-1)|^2 = 1$ for $\frac{3}{4} \le t \le \frac{5}{4}$

Many signals obey these properties.
The one shown is

$$g(t) = \begin{cases} \sin\left(\frac{\pi}{2}\sin^2\left(\pi\left(t+\frac{1}{4}\right)\right)\right) & -\frac{1}{4} \le t \le \frac{1}{4} \\ 1 & \frac{1}{4} \le t \le \frac{3}{4} \\ \sin\left(\frac{\pi}{2}\sin^2\left(\pi\left(\frac{9}{4}-t\right)\right)\right) & \frac{3}{4} \le t \le \frac{5}{4} \end{cases}$$



$g(t-n)$, $n = -2, \dots, 2$

LOT Basis

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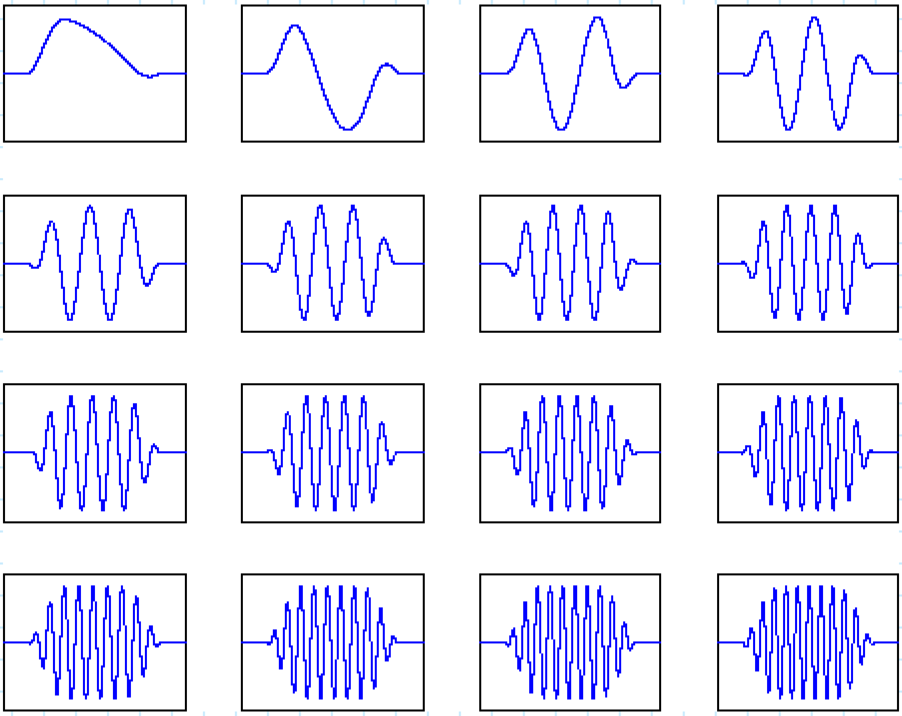
$$\phi_{n,k}(t) = g(t-n)\,\psi_k(t-n), \quad n \in \mathbb{Z}, \quad k \geq 0$$

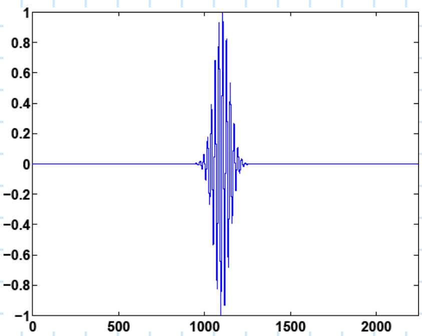
where $\psi_k(t-n) = \sqrt{2}\cos\left(\pi\left(k+\frac{1}{2}\right)(t-n)\right)$

LOT basis functions

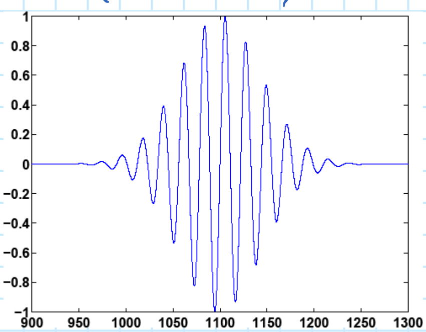


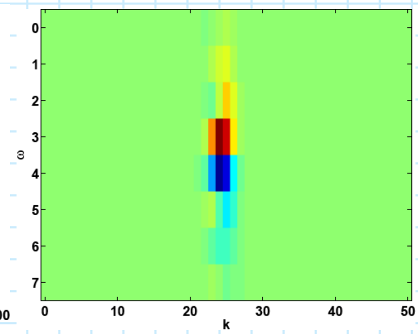
for $n=0$, $k=0,\dots,15$

$x(t)$



(zoomed)





LOT Details

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Time-domain / Frequency-domain

$\phi_{n,k}(t)$

Time-domain ? $\alpha_{k_0}(n) = \phi_{n,k_0}(t)$ $h[n] * \alpha_{k_0}(n)$

freq-domain ? $\beta_{n_0}(k) = \phi_{n_0,k}(t)$ $\beta_{n_0}(k) \cdot H(k)$

We need to show that for $x(t) \in L_2(\mathbb{R})$

$$x(t) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \langle x(t), \phi_{n,k}(t) \rangle \phi_{n,k}(t), \quad \text{for all } -\infty < t < \infty$$

we showed/argued that this worked for $\psi_{n,k}(t)$ but not $\phi_{n,k}(t)$

$$\langle x(t), \phi_{n,k}(t) \rangle = \int_{-\infty}^{\infty} x(t)\, \phi_{n,k}(t)\, dt = \int_{n-1/4}^{n+5/4} x(t)\, g(t-n)\, \psi_k(t-n)\, dt$$

$\psi_k(t)$ is the k^{th} cosine-IV basis vector extended to $t \in [-1/4, 5/4]$

$$\psi_k(t) = \sqrt{2} \cos\left(\left(k+\tfrac{1}{2}\right)\pi t\right) \quad \text{for } k = 0, 1, 2, \ldots$$

for LOT, we use $t \in [-1/4, 5/4]$

1 Mar 2017

Recall that

$$\psi_k(-t) = \psi(t), \quad \psi_k(t+1) = -\psi(-t+1)$$

$$\Updownarrow$$

$$\psi_k(2-t) = \psi_k(t)$$

LOT Details cont.

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for $n=0$

$$\langle x(t), \phi_{n,k}(t)\rangle = \int_{-1/4}^{5/4} x(t)\, g(t)\, \psi_k(t)\, dt =$$

$$\int_{-1/4}^{1/4} x(t) g(t)\, \psi_k(t)\, dt + \int_{1/4}^{3/4} x(t)\, \psi_k(t)\, dt + \int_{3/4}^{5/4} x(t) g(t)\, \psi_k(t)\, dt$$

(2)

$$\boxed{\psi_k(-t) = \psi_k(t)}$$

$$\int_{-1/4}^{1/4} \cdots = \int_{-1/4}^{0} x(t) g(t)\, \psi_k(t)\, dt + \int_{0}^{1/4} x(t) g(t)\, \psi_k(t)\, dt$$

$$= \int_0^{1/4} \big(x(t) g(t) + x(-t) g(-t)\big)\, \psi_k(t)\, dt \qquad (1)$$

similarly for the last term

$$\int_{3/4}^{5/4} \cdots\, dt = \int_{3/4}^{1} \big(x(t) g(t) - x(2-t) g(2-t)\big)\, \psi_k(t)\, dt \qquad (3)$$

combining (1), (2), (3)

$$\langle x(t), \phi_{0,k}(t)\rangle = \int_0^1 z_0(t)\, \psi_k(t)\, dt \quad \text{where}$$

$$z_0(t) = \begin{cases} x(t) g(t) + x(-t) g(-t) & 0 \le t \le 1/4 \\ x(t) & 1/4 \le t \le 3/4 \\ x(t) g(t) - x(2-t)\, g(2-t) & 3/4 \le t \le 1 \end{cases}$$

since $\{\psi_k(t)\}_k$ form an orthobasis for $[0,1]$, then for frame $n=0$

$$\sum_{k=0}^{\infty} \langle x(t), \phi_{0,k}(t)\rangle\, \phi_{0,k}(t) = \sum_{k=0}^{\infty} \langle z_0(t), \psi_k(t)\rangle_{L_2[0,1]}\, g(t)\, \psi_k(t)$$

$$= g(t) \Big(\sum_{k=0}^{\infty} \langle z_0, \psi_k\rangle_{L_2[0,1]}\, \psi_k(t) \Big)$$

nonzero on $(-1/4, 5/4)$

coefficients

defined on $[-2,2]$

$$= g(t)\, z_0(t) \quad \text{for} \quad t \in [0,1]$$

the sum reproduces $z_0(t)$ on the entire interval $[-1/4, 5/4]$

LOT Details III

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$$\sum_{k=0}^{\infty} \langle x(t), \phi_{0,k}(t)\rangle \, \phi_{0,k}(t) = g(t)\, z_0(t) \qquad \text{for } t \in \left[-\tfrac{1}{4}, \tfrac{5}{4}\right]$$

extended synthesis interval

repeating for $n=1$

$$\sum_{k=0}^{\infty} \langle x(t), \phi_{1,k}(t)\rangle \, \phi_{1,k}(t) = g(t-1)\, z_1(t) \qquad \text{for } t \in \left[\tfrac{3}{4}, \tfrac{9}{4}\right]$$

where

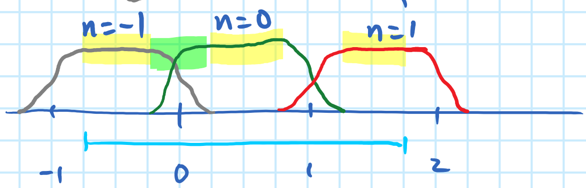
$$z_1(t) = \begin{cases} x(t)\,g(t-1) + x(2-t)\,g(1-t) & \tfrac{3}{4} \le t \le \tfrac{5}{4} \\ x(t) & \tfrac{5}{4} \le t \le \tfrac{7}{4} \\ x(t)\,g(t-1) - x(4-t)\,g(3-t) & \tfrac{7}{4} \le t \le \tfrac{9}{4} \end{cases}$$

and for $n=-1$

$$\sum_{k=0}^{\infty} \langle x(t), \phi_{-1,k}(t)\rangle \, \phi_{-1,k}(t) = g(t+1)\, z_{-1}(t) \qquad \text{for } t \in \left[-\tfrac{5}{4}, \tfrac{1}{4}\right]$$

where

$$z_{-1}(t) = \begin{cases} x(t)\,g(t+1) + x(-2-t)\,g(-t-1) & -\tfrac{5}{4} \le t \le -\tfrac{3}{4} \\ x(t) & -\tfrac{3}{4} \le t \le -\tfrac{1}{4} \\ x(t)\,g(t+1) - x(-t)\,g(1-t) & -\tfrac{1}{4} \le t \le \tfrac{1}{4} \end{cases}$$



adding these together

$$z_{-1}(t)\,g(t+1) + z_0(t)\,g(t) + z_1(t)\,g(t-1) = x(t)$$

true for the highlighted intervals above

for $t \in \left[-\tfrac{1}{4}, \tfrac{1}{4}\right]$

$$g(t+1)\,z_{-1}(t) + g(t)\,z_0(t) = g^2(t+1)\,x(t) - x(-t)\,g(1-t)\,g(t+1) + x(t)\,g^2(t) + x(-t)\,g(t)\,g(-t)$$

$$g^2(t) + g^2(t+1) = 1$$

$$= x(t) + 0$$

$g(t) = g(1-t)$ and

$g(-t) = g(t+1)$

The same holds true for all overlapping regions

LOT Completed

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Therefore

LOT basis

$$\sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \langle x(t), \phi_{n,k}(t) \rangle \phi_{n,k}(t) = \sum_{n=-\infty}^{\infty} g(t-n) z_n(t) = x(t) \quad \forall\, t \in \mathbb{R}$$