Haar wavelets

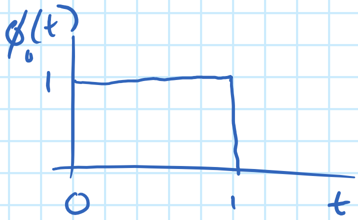
Wednesday, March 1, 2017 10:03 AM

The Haar wavelet basis for $L_2(\mathbb{R})$ breaks down a signal by looking at the difference between piece-wise constant approximations at different scales. This is the easiest wavelet transform to understand...

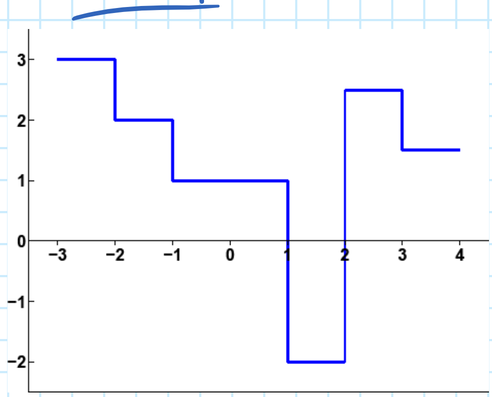
Let V_0 be the space of signals that are piecewise constant between integers

set

$$\phi_0(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$



example



$\{\phi_0(t-n),\ n \in \mathbb{Z}\}$ is an orthobasis for V_0

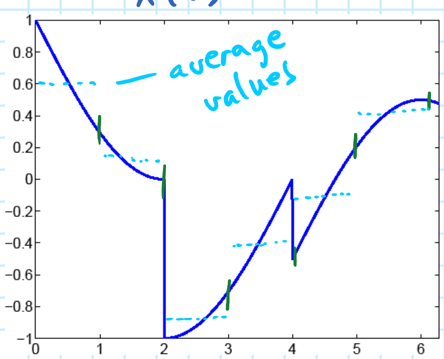
define $\phi_{0,n}(t) = \phi_0(t-n)$

we can project an arbitrary $x(t)$ onto V_0

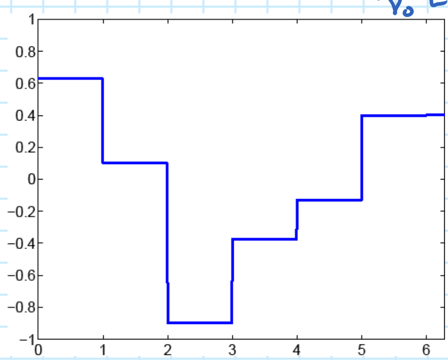
$$\hat{x}_0(t) = \sum_{n=-\infty}^{\infty} \langle x(t), \phi_{0,n}(t) \rangle \, \phi_{0,n}(t)$$

we will introduce some notation to make this more compact.

$x(t)$



$\hat{x}_0(t) = P_{V_0}[x(t)]$



$$\hat{x}_0(t) = P_{V_0}[x(t)]$$

projection operator

This returns the signal in V_0 that is closest to $x(t)$

Haar wavelets - $\mathbf{V}_j$

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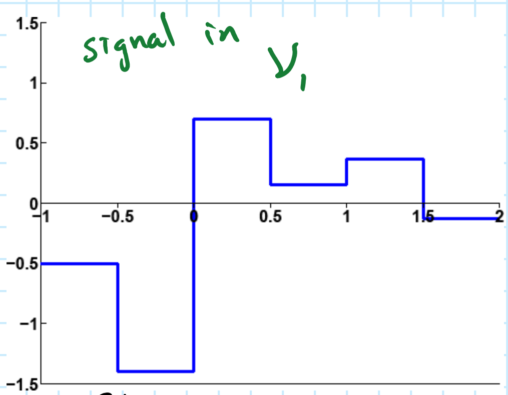
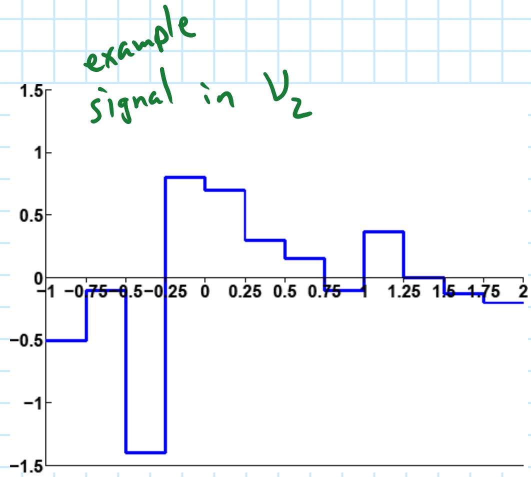
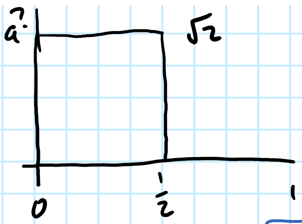
Let V_j be the space of piecewise-constant functions at a higher resolution. V_j contains signals that are constant on intervals of length 2^{-j}





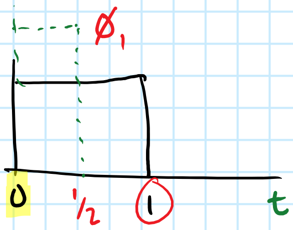


$$\int_0^{1/2} a^2\, dt = 1$$

$$a^2 t \Big|_0^{1/2} = \frac{1}{2} a^2 = 1 \Rightarrow a = \sqrt{2}$$

$$\boxed{\phi_j(t) = 2^{j/2}\, \phi_0(2^j t)}$$ basis for V_j

We obtain an orthobasis for V_j by contracting the basis for V_0 by a dyadic factor



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j = an index

$i = \sqrt{-1}$

$$\phi_{j,n}(t) = \phi_j(t - 2^j n) = 2^{j/2}\, \phi_0(2^j t - n)$$

$\phi_{j,0}(t - 2^j n)$

This forms an orthobasis for V_j

Haar Scaling Spaces

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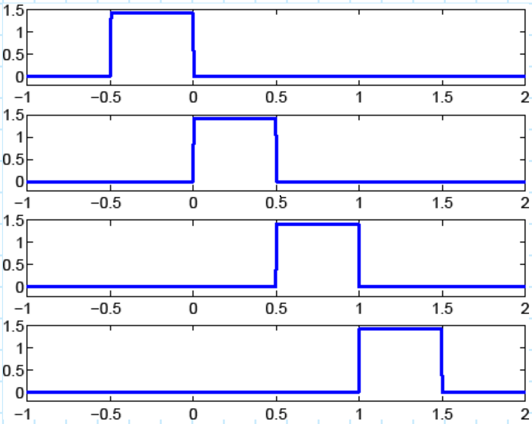
$\phi_{j,n}(t) \quad n = -1, \dots, 2$

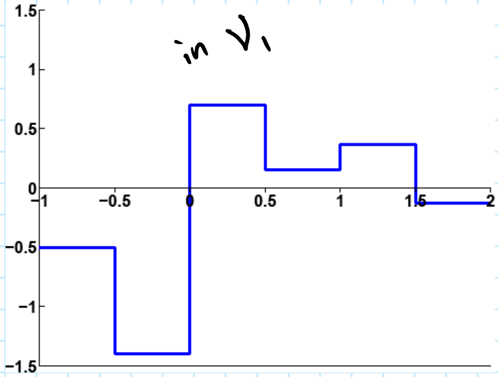


$\phi_{1,-1}(t)$

$\phi_{1,0}(t)$

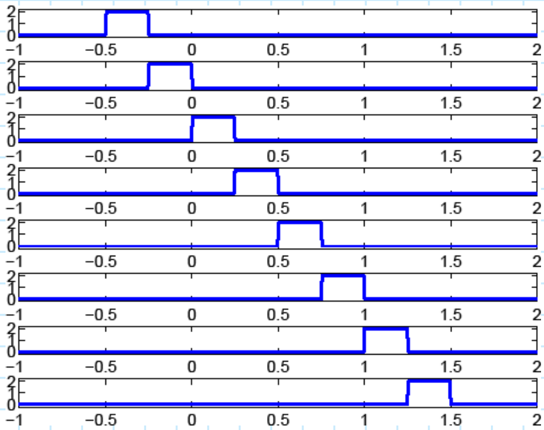
$\phi_{1,1}(t)$

$\phi_{1,2}(t)$

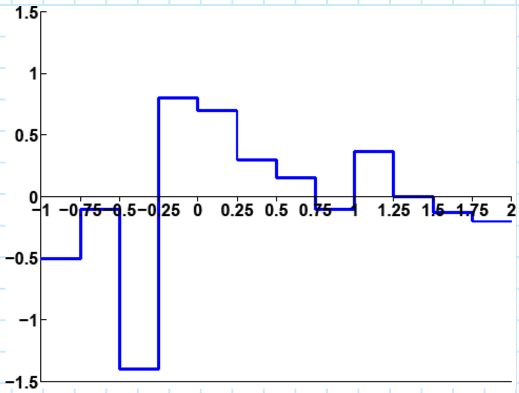


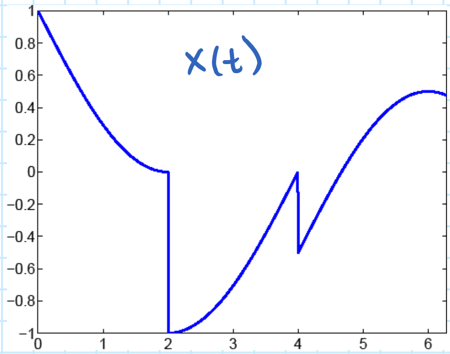
$\phi_{2,n}$ for $n = -2, \dots, 5$

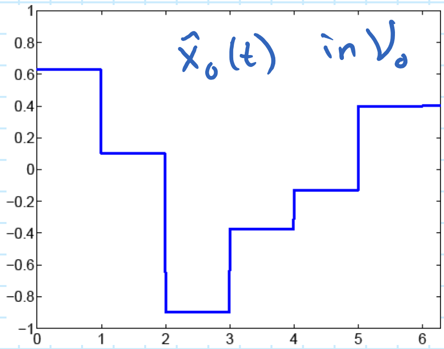
Waveform in V_2

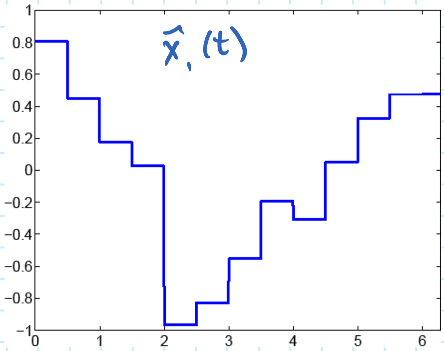


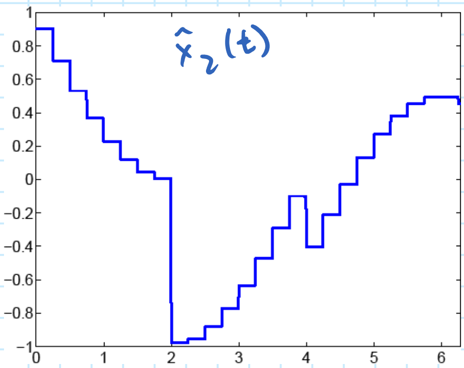
n
-2
-1
0
1
2
3
4
5

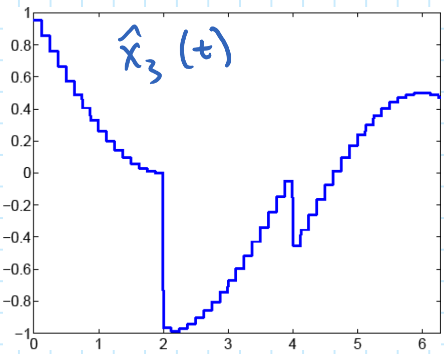


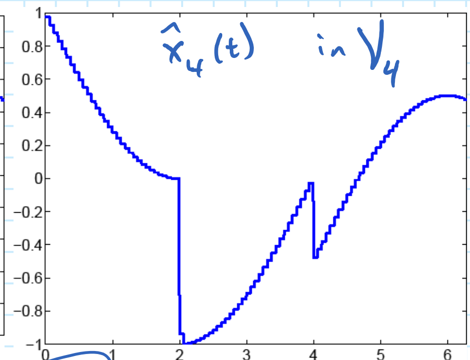












$$\boxed{\lim_{j \to \infty} \hat{x}_j(t) = \lim_{j \to \infty} P_{V_j}[x(t)] = x(t)}$$

Wavelets

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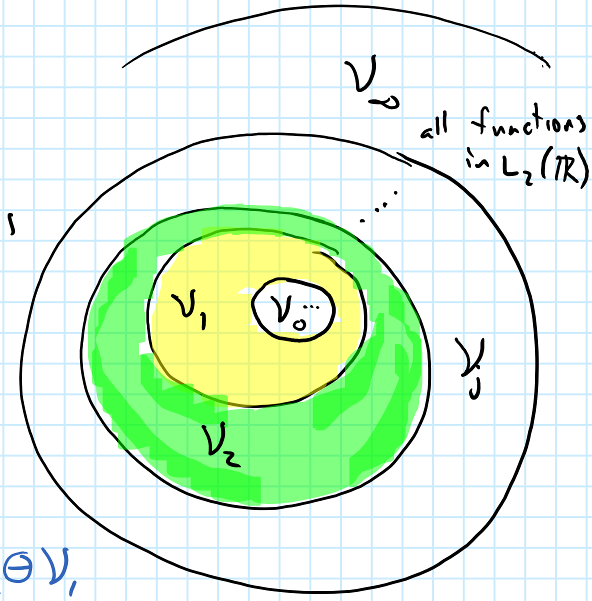
$\phi_{j,n}(t)$ ← scaling functions

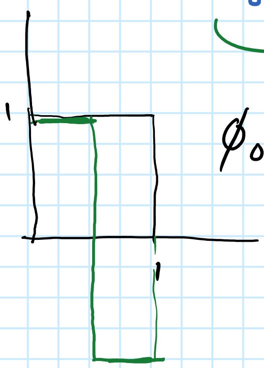


The Haar wavelets represent the differences between approximations in V_j and V_{j-1}

$$W_0 = V_1 \ominus V_0 \qquad W_1 = V_2 \ominus V_1$$

W_0 signals are constant over intervals of length ½ but are orthogonal to everything in V_0

They must be zero-mean between integers



$\phi_{0,0}(t)$

$$\psi_0(t) = \begin{cases} 1 & 0 \le t < \tfrac{1}{2} \\ -1 & \tfrac{1}{2} \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$\psi_{0,n}(t)$ form an orthobasis for W_0

$$\psi_{j,n}(t) = 2^{j/2}\,\psi_0(2^j t - n)$$ ← orthobasis for $W_{j,n}$ for $n \in \mathbb{Z}$

These give us a multi-scale approximation

$$x(t) = \underbrace{\underbrace{\underbrace{P_{V_0}(x(t)) + P_{W_0}(x(t))}_{P_{V_1}(x(t))} + P_{W_1}(x(t))}_{P_{V_2}(x(t))} + P_{W_2}(x(t))}_{P_{V_3}(x(t))} + \dots$$

Haar wavelet system

$$x(t) = \overbrace{\sum_{n=-\infty}^{\infty} \langle x, \phi_{0,n}\rangle\, \phi_{0,n}(t)}^{P_{V_0}(x(t))} + \sum_{j=0}^{\infty} \underbrace{\sum_{n=-\infty}^{\infty} \langle x, \psi_{j,n}\rangle\, \psi_{j,n}(t)}_{P_{W_j}[x(t)]}$$

Haar wavelet plots

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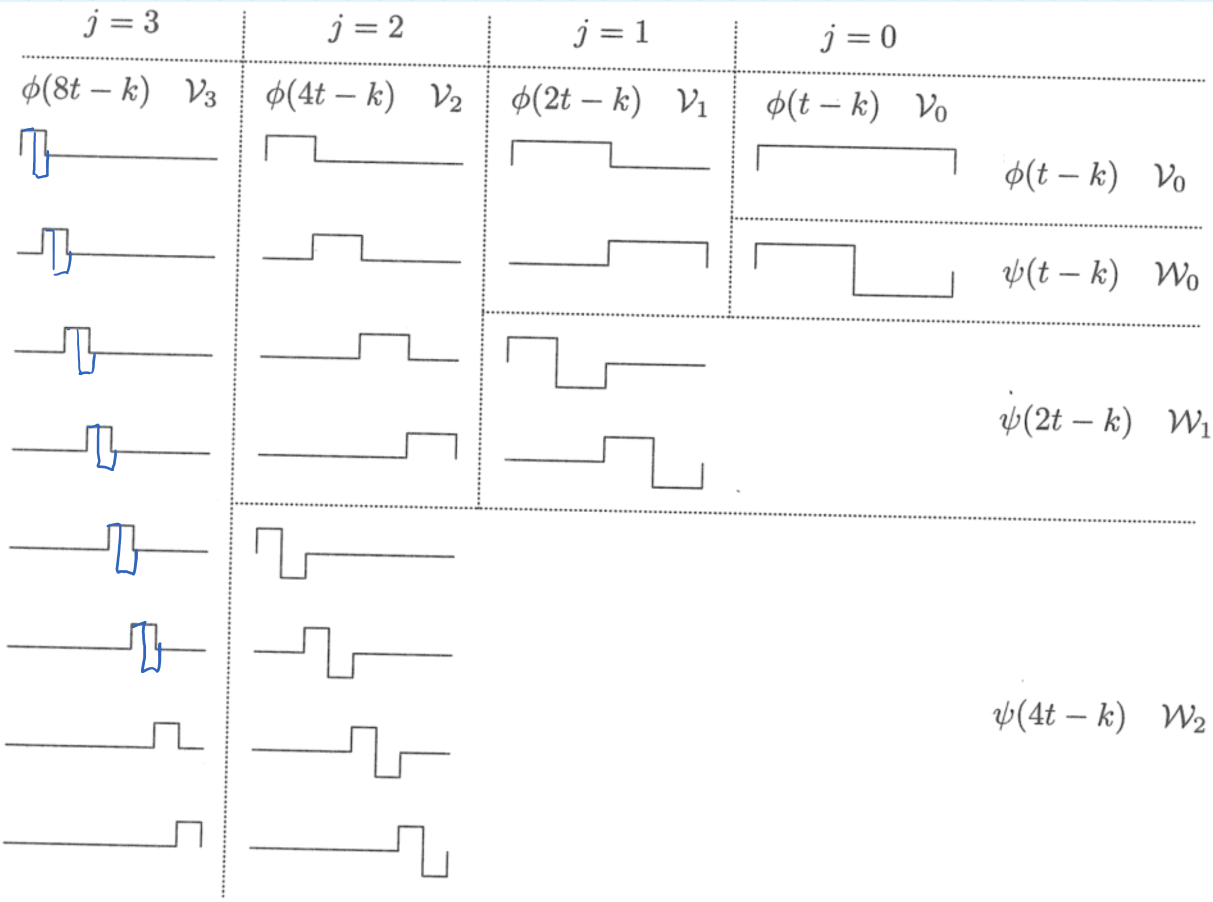

$$\mathcal{V}_3 = \mathcal{V}_0 \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2$$

Nomenclature

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- We call the V_j the scaling spaces or approximation spaces
- The $\phi_{j,n}(t)$ for $n \in \mathbb{Z}$ form an orthobasis for V_j and they are called scaling functions at scale j
- The W_j are called wavelet spaces or detail spaces
- The expansion coefficients that specify the approximation $\hat{x}_j(t)$ in terms of $\phi_{j,n}(t)$ are called scaling coefficients

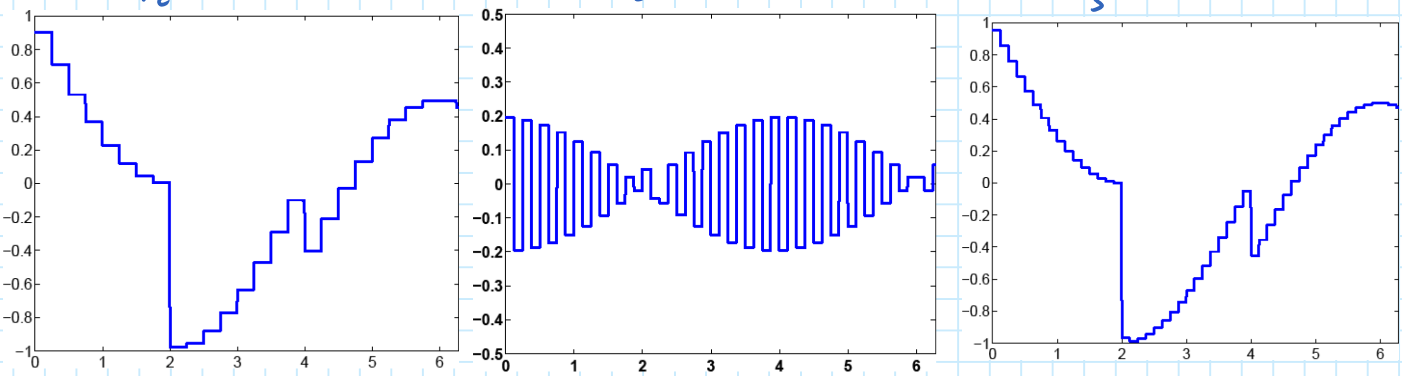
$$\hat{x}_j(t) = \sum_n s_{j,n}\, \phi_{j,n}(t) \qquad s_{j,n} = \langle x, \phi_{j,n} \rangle$$

- Correspondingly, the wavelet coefficients at scale j are

$$w_{j,n} = \langle x, \psi_{j,n} \rangle$$

and $$P_{W_j}[x(t)] = \sum_{n=-\infty}^{\infty} w_{j,n}\, \psi_{j,n}(t)$$

$$P_{V_2}[x(t)] \quad + \quad P_{W_2}[x(t)] \quad = \quad P_{V_3}[x(t)]$$



"detail space"

We can express $P_{V_3}[x(t)]$ in terms of $\phi_{3,n}(t)$ **or** in terms of $\phi_{2,n}(t)$ and $\psi_{2,n}(t)$

In general, we can pick some scale as the starting approximation and then add detail with the wavelets using as many scales as needed → our approximation could be written in terms of the coefficients $\{s_{0,n}, w_{0,n}, w_{1,n}, \ldots, w_{J-1,n}\}$

(same as $\{s_{j,n}\}$)

Toward Filterbanks

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We can compute the approximations efficiently

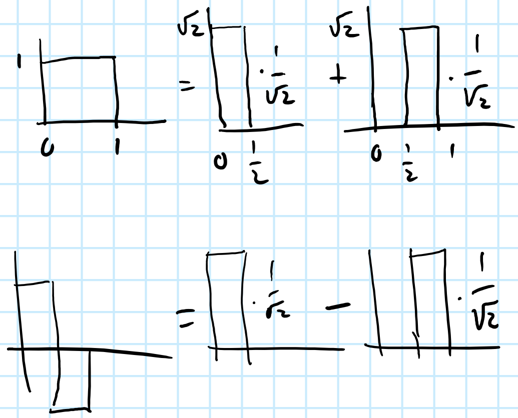
$$W_0 = V_1 \ominus V_0 \quad \Rightarrow \quad V_1 = V_0 \oplus W_0$$

$$\phi_0(t) = \frac{1}{\sqrt{2}}\left(\phi_1(t) + \phi_1(t - 1/2)\right)$$

$$= \frac{1}{\sqrt{2}}\left(\phi_{1,0}(t) + \phi_{1,1}(t)\right)$$

and

$$\psi_0(t) = \frac{1}{\sqrt{2}}\left(\phi_{1,0}(t) - \phi_{1,1}(t)\right)$$



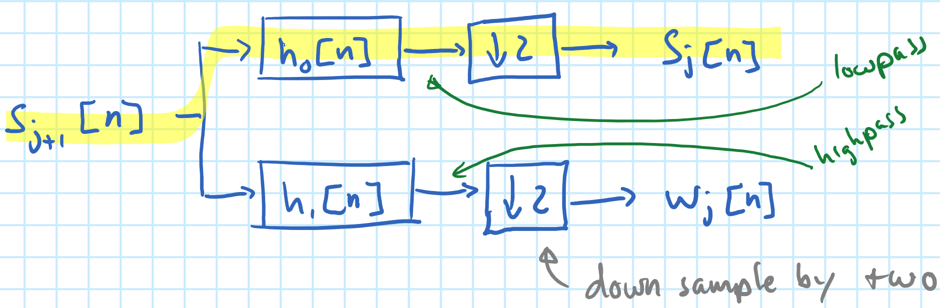
$$\phi_{j,n}(t) = \frac{1}{\sqrt{2}}\left(\phi_{j+1,2n}(t) + \phi_{j+1,2n+1}(t)\right)$$

$$\psi_{j,n}(t) = \frac{1}{\sqrt{2}}\left(\phi_{j+1,2n}(t) - \phi_{j+1,2n+1}(t)\right)$$

If we have $s_{j+1,n}$ at scale $j+1$,

$$s_{j,n} = \langle x, \phi_{j,n} \rangle = \frac{1}{\sqrt{2}}\left(s_{j+1,2n} + s_{j+1,2n+1}\right)$$

$$w_{j,n} = \langle x, \psi_{j,n} \rangle = \frac{1}{\sqrt{2}}\left(s_{j+1,2n} - s_{j+1,2n+1}\right)$$



for Haar wavelets