Best Linear Unbiased Estimator

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$y = Ax_0 + e$ where $e \in \mathbb{R}^M$ is random

$E\{e\} = \underline{0}$, $E\{ee^T\} = R$

What is the best statistical estimate of x_0?

1) Linearity → for some matrix L. $(L \in \mathbb{R}^{N \times M})$

$\hat{x} = Ly$

2) Unbiased → $E\{\hat{x}\} = x_0$

$E\{\hat{x} - x_0\} = \underline{0}$

3) "best" implies $E\{\|\hat{x} - x_0\|_2^2\}$ is minimized

$$\hat{x} = Ly = L(Ax_0 + e) = LAx_0 + Le$$

unbiased → $E\{\hat{x} - x_0\} = LAx_0 - \overset{0}{\cancel{E\{Le\}}} - x_0 = 0$

$$\boxed{LAx_0 = x_0}$$

L is the left inverse of A ⇒ $LA = I$

"best" → $E\{\|x_0 - \hat{x}\|_2^2\} = E\{\|x_0 - LAx_0 - Le\|_2^2\}$

$= E\{\|Le\|_2^2\}$ ← find L to minimize this

we want

$$\min_{\substack{L:\,N\times M \\ LA = I}} E\{\|Le\|_2^2\} = \min_{\substack{L:\,N\times M \\ LA = I}} \operatorname{trace}(LRL^T)$$

The solution is $\boxed{L_0 = (A^T R^{-1} A)^{-1} A^T R^{-1}}$ ← let's check this

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$$L_0 A = (A^T R^{-1} A)^{-1} (A^T R^{-1} A) = I \quad \checkmark$$

we will show that for any other L

$$\operatorname{trace}(L R L^T) \geq \operatorname{trace}(L_0 R L_0^T)$$

let $L = L_0 + (L - L_0)$

$$\operatorname{trace}(LRL^T) = \operatorname{trace}(L_0 R L_0^T) + \operatorname{trace}(L_0 R (L-L_0)^T)$$
$$+ \operatorname{trace}((L-L_0)^T R L_0^T) + \operatorname{trace}((L-L_0) R (L-L_0)^T)$$

$$R L_0^T = R R^{-1} A (A^T R^{-1} A)^{-1} = A (A^T R^{-1} A)^{-1}$$

$$(L-L_0) R L_0^T = (L - L_0) A (A^T R^{-1} A)^{-1}$$
$$= \overset{0}{\cancel{(I - I)}} (A^T R^{-1} A)^{-1} = 0$$

$$\operatorname{trace}(LRL^T) = \operatorname{trace}(L_0 R L_0^T) + \underbrace{\operatorname{trace}((L-L_0) R (L-L_0)^T)}$$

R is positive semi-def. so this term ≥ 0

$$\operatorname{trace}(LRL^T) \geq \operatorname{trace}(L_0 R L_0^T)$$

for any L satisfying $LA = I$

Best Linear Unbiased Estimator (BLUE)

$$y = A x_0 + e \qquad E\{ee^T\} = R \qquad E\{e\} = \underline{0}$$

$$\boxed{\hat{x}_{blue} = (A^T R^{-1} A)^{-1} A^T R^{-1} y}$$

$$L_0 R L_0^T = (A^T R^{-1} A)^{-1}$$

$$E\{\|x_0 - \hat{x}_{blue}\|_2^2\} = \text{trace}\left((A^T R^{-1} A)^{-1}\right)$$

$$= \text{sum of eigenvalues of } (A^T R^{-1} A)^{-1}$$

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example

$$y = A x_0 + e \qquad A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad y = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \quad R = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

$$R^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\hat{x}_{blue} = \left(\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \cancel{\frac{1}{5}} \right)^{-1} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \cdot \cancel{\frac{1}{5}} \cdot \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$$= \left(\begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$$= \frac{1}{7} \cdot 20 = \frac{20}{7}$$ — our best estimate of x_0

$$E\{\|x_0 - \hat{x}_{blue}\|_2^2\} = \frac{1}{7}$$

Weighted Least Squares

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what if some measurements that are more reliable than others?

↳ BLUE can deal with this

but another way of solving is ...

let $r = y - Ax$ ← residual

minimize $\|Wr\|_2^2 = \|Wy - WAx\|_2^2$

weighting matrix W

(it is usually diagonal)

$$\boxed{\hat{x}_{wls} = (WA)^{\dagger} Wy}$$