Kalman Filters

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minimize $E\{e^2[n]\}$ ← like LMS except.

1. KF does not observe $d[n]$ directly
2. KF works in highly non-stationary environments

The KF is state-based

We have some system model and keep track of the system state

example: we have a car moving at a constant speed

state: car's position & speed

$$\underline{x}(n) = \begin{pmatrix} p(n) \\ s(n) \end{pmatrix} \begin{matrix} \leftarrow \text{position} \\ \leftarrow \text{speed} \end{matrix}$$

at time n+1

$$\underline{x}(n+1) = \begin{pmatrix} p(n) + \Delta t \cdot s(n) \\ s(n) \end{pmatrix}$$

we can write this as

$$\boxed{\underline{x}(n+1) = A\,\underline{x}(n) + \underline{w}(n)} \quad \text{where } A = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix}$$

we usually have noise

– this noise is the driver – he can adjust the speed

$$\underline{w}(n) = \begin{pmatrix} 0 \\ \text{something} \end{pmatrix}$$

measurement noise

$$y(n) = x_1(n) + v(n)$$

we measure position but not speed and we have measurement noise

$$y(n) = C\,\underline{x}(n) + v(n)$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Approximation Error

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These are mutually orthogonal

$$\hat{x}_{trunc} - \hat{x}_{pinv} = \underbrace{\sum_{k=R'+1}^{R} -\langle x, v_k\rangle v_k}_{\text{approximation error}} + \underbrace{\sum_{r=1}^{R'} \frac{1}{\sigma_r}\langle e, u_r\rangle v_r}_{\text{noise error}}$$

this means that

$$\left\| \hat{x}_{trunc} - \hat{x}_{pinv} \right\|_2^2 = \| \text{Approx. error} \|_2^2 + \| \text{noise error} \|_2^2$$

$$= \sum_{k=R'+1}^{R} |\langle x, v_r\rangle|^2 + \sum_{r=1}^{R'} \frac{1}{\sigma_r^2} |\langle e, u_r\rangle|^2$$

Assuming, as before, that $e[m] \sim N(0, \nu^2)$

$$E\left[\| \text{Noise error} \|_2^2 \right] = \nu^2 \sum_{r=1}^{R'} \frac{1}{\sigma_r^2}$$

Aside

$$\|x\|_2^2 = \langle x, x\rangle$$

$$\sum_{r=1}^{R} \underbrace{\frac{1}{\sigma_r}\langle e, u_r\rangle}_{\text{scalar}} v_r$$ — the error due to noise $\hat{x}_{trunc} - \hat{x}_{pinv}$

$$\gamma_r = \frac{1}{\sigma_r}\langle e, u_r\rangle$$

$$\sum_r \gamma_r \underline{v}_r \Rightarrow \|\text{error}\|_2^2 = \left\langle \sum_r \gamma_r \underline{v}_r , \sum_k \gamma_k \underline{v}_k \right\rangle$$

$$= \sum_r \gamma_r \left\langle v_r, \sum_k \gamma_k \underline{v}_k \right\rangle$$

$$\sum_r \gamma_r \sum_k \gamma_k V_k^T V_r$$

$$\langle v_r, v_k\rangle = \begin{cases} 1 & r = k \\ 0 & \text{else} \end{cases}$$

$$\frac{\|u_R\|_2}{\sigma_R} \cdot \|u_R\|^2$$

normalization factor

$$\langle v_r, v_k \rangle = \begin{cases} \cdot & \cdot \\ 0 & \text{else} \end{cases}$$

$$= \sum_r \gamma_r^2 = \sum_r \frac{1}{\sigma_r^2} \langle e, u_r \rangle^2$$

if $e = u_R$

$$= \frac{1}{\sigma_R^2} \langle u_R, u_R \rangle^2$$

Kalman Filter Notation

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State equation

$$\underline{x}(n+1) = A(n+1, n)\,\underline{x}(n) + B(n)\,\underline{w}(n)$$

goal is to find $x(n)$

Observation equation

$$\underline{y}(n) = C(n)\,\underline{x}(n) + D(n)\,\underline{v}(n)$$

transforms state to observation

$\underline{v}(n)$ - observation noise

(for this development, assume B & D are identity matrices)

$$E\{\underline{v}(n)\,\underline{v}^T(n)\} = Q_v\,\delta(n-k)$$

each time step is uncorrelated with all others but at a particular time, there may be internal correlations (in Q_v)

$\underline{w}(n)$ - process noise

$$E\{\underline{w}(n)\,\underline{w}^T(n)\} = Q_w\,\delta(n-k)$$

Let $\boxed{\hat{\underline{x}}(n|i)}$ be the unbiased minimum mean-square estimate of $\underline{x}(n)$ at time n given all the measurements up to and including time i

Let $\boxed{\underline{e}(n|i)}$ be the corresponding state estimation error

$$\underline{e}(n|i) = \underline{x}(n) - \hat{\underline{x}}(n|i)$$

and

$$\boxed{P(n|i) = E\{\underline{e}(n|i)\,\underline{e}^T(n|i)\}}$$

is the error covariance matrix

Goal: minimize MSE

$$E\{\|e(n|n)\|^2\} = Tr\{P(n|n)\} = \sum_{k=0}^{P-1} E\{e_k^2(n|n)\}$$

solution

$$\hat{\underline{x}}(n|n) = \underbrace{A(n,n-1)\,\hat{\underline{x}}(n-1|n-1)}_{\text{normal state transition}} + K(n)\left[\underbrace{\underline{y}(n)}_{\text{actual observation}} - \underbrace{C(n)\,A(n,n-1)\,\hat{\underline{x}}(n-1|n-1)}_{\text{expected observation}}\right]$$

transforms error into a state update

Kalman Filter Development

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Given $\hat{\underline{x}}(n-1|n-1)$ and $P(n-1|n-1)$, when $y(n)$ becomes available, we want to find the unbiased estimate $\hat{x}(n|n)$ that minimizes $\mathrm{Tr}\{P(n|n)\}$

Proceed in two steps.

1) Given $\hat{\underline{x}}(n-1|n-1)$, find $\hat{\underline{x}}(n|n-1)$ ← "Prediction step"
   - use state transition equations

2) Given $y(n)$ and $\hat{\underline{x}}(n|n-1)$, find $\hat{\underline{x}}(n|n)$ — our estimate of $\underline{x}(n)$

Prediction step detail

Since $\underline{w}(n)$ is zero mean

$$\hat{\underline{x}}(n|n-1) = A(n,n-1)\,\hat{\underline{x}}(n-1|n-1)$$

$$\begin{aligned}
\underline{e}(n|n-1) &= \underline{x}(n) - \hat{\underline{x}}(n|n-1) \\
&= A(n,n-1)\,\underline{x}(n-1) + \underline{w}(n) - A(n,n-1)\,\hat{\underline{x}}(n-1|n-1) \\
&= A(n,n-1)\underbrace{\left[\underline{x}(n-1) - \hat{\underline{x}}(n-1|n-1)\right]}_{\underline{e}(n-1|n-1)} + \underline{w}(n)
\end{aligned}$$

if $\hat{\underline{x}}(n-1|n-1)$ is unbiased then

$$E\{\underline{e}(n|n-1)\} = A(n,n-1)\,\underbrace{\cancel{E\{\underline{e}(n-1|n-1)\}}}_{\hat{x}(n-1|n-1)\text{ is unbiased}}^{0} + \underbrace{\cancel{E\{\underline{w}(n)\}}}_{w(n)\text{ is zero mean}}^{0}$$

$= 0$ so $\hat{\underline{x}}(n|n-1)$ is also unbiased

Kalman Development cont.

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$$P(n|n-1) = E\{\underline{e}(n|n-1)\,\underline{e}^T(n|n-1)\}$$

$$= E\left\{\left(A(n,n-1)\,\underline{e}(n-1|n-1) + \underline{w}(n)\right)\left(A(n,n-1)\,\underline{e}(n-1|n-1) + \underline{w}(n)\right)^T\right\}$$

since $\underline{e}(n-1|n-1)$ is uncorrelated with $\underline{w}(n)$

$$\boxed{P(n|n-1) = A(n,n-1)\,P(n-1|n-1)\,A^T(n,n-1) + Q_w(n)}$$

Ricatti Equation

$$\hat{x}(n|n-1) = A(n,n-1)\,\hat{x}(n-1|n-1)$$

step 2 observation

we will use a linear estimator

$$\underline{\hat{x}(n|n) = K'(n)\,\underline{\hat{x}}(n|n-1) + K(n)\,\underline{y}(n)}$$

we require $\underline{\hat{x}}(n|n)$ to be unbiased $\Rightarrow$ $\boxed{E\{e(n|n)\} = 0}$

and we want to choose the K' & K

to minimize $E\{\|e(n|n)\|^2\}$

$$e(n|n) = x(n) - K'(n)\,\underline{\hat{x}}(n|n-1) - K(n)\,\underline{y}(n)$$

$$= \underline{x}(n) - K'(n)\left[\underline{x}(n) - \underline{e}(n|n-1)\right] - K(n)\left[C(n)\,\underline{x}(n) + \underline{v}(n)\right]$$

$$= \underbrace{\left[I - K'(n) - K(n)C(n)\right]}_{\text{for unbiased, we must have } I - K'(n) - K(n)C(n) = 0}\underline{x}(n) + K'(n)\underbrace{e(n|n-1)}_{E\{\ \}=0} - K(n)\underbrace{\underline{V}(n)}_{E\{\ \}=0}$$

because $E\{e(n|n-1)\} = 0$

because $E\{\underline{v}(n)\} = 0$

The Kalman Gain

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$K'(n) = I - K(n)C(n)$ for our estimate to be unbiased

$$\hat{x}(n|n) = [I - K(n)C(n)]\,\hat{\underline{x}}(n|n-1) + K(n)\,y(n)$$

$$\boxed{\hat{x}(n|n) = \hat{x}(n|n-1) + \underline{\underline{K(n)}}\left[y(n) - C(n)\,\hat{x}(n|n-1)\right]}$$

need to find $K(n)$ now

guarantees unbiasedness

$$\underline{e}(n|n) = [I - K(n)C(n)]\,\underline{e}(n|n-1) - K(n)\,\underline{V}(n)$$

uncorrelated with $w(n)$
uncorrelated with $y(n-1)$
with $\underline{x}(n)$ and
" with $\hat{x}(n|n-1)$

$$P(n|n) = E\{e(n|n)\,e^T(n|n)\}$$

$$= [I - K(n)C(n)]\,P(n|n-1)\,[I - K(n)C(n)]^T + K(n)\,Q_v(n)\,K^T(n)$$

To minimize $tr\{P(n|n)\}$ w.r.t. $K(n)$.

$$\frac{\partial}{\partial K} tr\{P(n|n)\} = 0$$

$$\frac{\partial}{\partial K} tr(KA) = A^T$$

$$\frac{\partial}{\partial K} tr(KAK') = 2KA$$

yielding ...

$$[K(n)C(n) - I]\,P(n|n-1)\,C^T(n) + K(n)\,Q_v(n) = 0$$

solving for $k(n)$

$$\boxed{K(n) = P(n|n-1)\,C^T(n)\left[C(n)P(n|n-1)C^T(n) + Q_v(n)\right]^{-1}}$$

Updating P(n|n)

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from the previous page

$$K(n) = P(n|n-1)\, C^T(n) \left[C(n) P(n|n-1)\, C^T(n) + Q_v(n) \right]^{-1}$$

$$\left[K(n) C(n) - I \right] P(n|n-1)\, C^T(n) + K(n)\, Q_v(n) = 0$$

$$P(n|n) = \left[I - K(n) C(n) \right] P(n|n-1) \left[I - K(n) C(n) \right]^T + K(n)\, Q_v(n)\, K^T(n)$$

$$= \left[I - K(n) C(n) \right] P(n|n-1) + \left\{ \underbrace{\left[K(n) C(n) - I \right] P(n|n-1)\, C^T(n) + K(n)\, Q_v(n)}_{=0} \right\} K^T(n)$$

$$\boxed{P(n|n) = \left[I - K(n) C(n) \right] P(n|n-1)}$$

Summary

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State equation : $\underline{x}(n) = A(n,n-1)\,\underline{x}(n-1) + \underline{w}(n)$

observation equation : $\underline{y}(n) = C(n)\,\underline{x}(n) + \underline{v}(n)$

Noise statistics : $E\{\underline{w}(n)\,\underline{w}^T(n)\} = Q_w$

$E\{\underline{v}(n)\,\underline{v}^T(n)\} = Q_v$

Initialization : $\hat{\underline{x}}(0|0) = E\{\underline{x}(0)\}$

$P(0|0) = E\{(\underline{x}(0) - E\{\underline{x}(0)\})(\underline{x}(0) - E\{\underline{x}(0)\})^T\}$

Computation : for $n = 1, \ldots$

$$\hat{\underline{x}}(n|n-1) = A(n,n-1)\,\hat{\underline{x}}(n-1|n-1)$$

$$P(n|n-1) = A(n,n-1)\,P(n-1|n-1)\,A^T(n,n-1) + Q_w$$

$$K(n) = P(n|n-1)\,C^T(n)\left[C(n)\,P(n|n-1)\,C^T(n) + Q_v\right]^{-1}$$

$$\hat{\underline{x}}(n|n) = \hat{\underline{x}}(n|n-1) + K(n)\left[\underline{y}(n) - C(n)\,\hat{\underline{x}}(n|n-1)\right]$$

$$P(n|n) = \left[I - K(n)\,C(n)\right]P(n|n-1)$$

comments :
- often we know Q_v based on characterization of our sensors
- Q_w is often hard to measure, overestimating is better than underestimating it.
- A large Q_w can compensate for a poor model
- We often tune the filter by tweaking Q_w & Q_v
- Make $P(0|0)$ very large if you are uncertain about $x(0|0)$