GEORGIA INSTITUTE OF TECHNOLOGY
School of Electrical and Computer Engineering

Pre-test

Due: January 17, 2018  Course: ECE 6254

Name: ________________________________

Last, __________________ First

• Open book/internet.
• No time limit.
• The test is worth 100 points. There are ten questions, and each one is worth 10 points. In multi-part questions, each part will be weighted equally.
• All work should be performed on the test itself. If more space is needed, use the backs of the pages.
• This test will be conducted under the rules and guidelines of the Georgia Tech Honor Code and no cheating will be tolerated (i.e., no discussing the test with other students).
• Make sure to look at the question titles, they will sometimes provide valuable hints.
• Please contact Prof. Anderson (in person or via email) directly with any questions if you are unclear what a question is asking.
Problem 1: Random variables. Suppose that two independent random variables $X$ and $Y$ are distributed according to

$$X \sim \text{Uniform}(0, 3) \quad Y \sim \text{Uniform}(1, 4)$$

What is the probability that $X < Y$.

Problem 2: Independence: Suppose that the probability that $A$ occurs is 0.4 and the probability that both $A$ and $B$ occur is 0.25. If $A$ and $B$ are independent events, what is the probability that neither $A$ nor $B$ occur?
Problem 3: Conditional probability density functions and derived distributions:
Suppose that $X$ and $Y$ have joint pdf given by

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-2x-y} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$ 

(a) What are the marginal probability density functions for $X$ and $Y$?

(b) What is the conditional probability density function $f_{X|Y}(x|y)$?
Problem 4: The median and the cumulative distribution function: Let $M$ be the number of miles your electric car can drive without running out of electricity, and suppose that $M$ has probability density function given by

$$f_M(m) = \begin{cases} \frac{e^{-m/301}}{301} & m \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

What is the median range for your car? That is, how far can you drive before there is a 50% chance that your battery runs out?
Problem 5: Bayes rule and normal random variables: Let \( X \) be the score of a randomly selected student in this class on an exam. Let \( Y \) denote the mean score of the class. If we knew \( Y \), suppose that a reasonable model for \( X \) would be that \( X \) is normal with mean \( Y \) and variance 20, that is,\(^1\)

\[
f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}20} e^{-(x-y)^2/40}
\]

Suppose that I think that the mean for the exam is probably between 70 and 85, i.e.,

\[
f_Y(y) = \begin{cases} 
\frac{1}{15} & 70 \leq y \leq 85 \\
0 & \text{otherwise.}
\end{cases}
\]

Suppose that I randomly select a quiz, grade it, and it turns out to receive a 100. Using Bayes rule, how should I update my distribution for \( Y \)? That is, what is \( f_{Y|X}(y|X = 100) \)? (Simplify as much as possible, but you may leave your answer in terms of the standard normal cumulative distribution function \( \Phi \) if you wish.)

\(^1\)Note that this pdf technically allows some probability of scores above 100 and below 0. You should just ignore this for now.
Problem 6: Joint probability density functions: The correlation coefficient \( \rho(X, Y) \) between a pair of random variables \( X \) and \( Y \) is given by

\[
\rho(X, Y) = \frac{E[(X - E[X]) \cdot (Y - E[Y])]}{\sigma_X \sigma_Y}.
\]

Suppose that \( X \) and \( Y \) have joint pdf given by

\[
f_{X,Y}(x, y) = \begin{cases} 
\frac{3}{4}x^2(1-y) & 0 \leq x \leq 2, 0 \leq y \leq 1 \\
0 & \text{else.}
\end{cases}
\]

for any \( x, y \). What is \( \rho \) in this case?
Problem 7: Linear Algebra: Pythagoras?

(a) Under what conditions on \( \mathbf{x} \) and \( \mathbf{y} \) is it true that
\[
\|\mathbf{x} + \mathbf{y}\|_2^2 = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2
\]

(b) Under what conditions on \( \mathbf{x} \) and \( \mathbf{y} \) is it true that
\[
\|\mathbf{x} + \mathbf{y}\|_2 = \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2
\]
Problem 8: Singular value decomposition. Let

\[
A = \begin{bmatrix}
-2 & 2 & 2 & -2 & 0 \\
2 & -2 & -2 & 2 & 0 \\
0 & 0 & 0 & 0 & 2
\end{bmatrix}
\]

(a) What is rank(\(A\))?

(b) Using Python or MATLAB (or whatever) find the singular value decomposition of \(A\). That is, find matrices \(U, \Sigma, V\) such that

\[
A = U \Sigma V^T
\]

and \(U^T U = I\), \(V^T V = I\), and \(\Sigma\) has non-negative entries along its diagonal and is zero elsewhere.
(c) Describe, in words, the column space (or range) of $A$:

$$\text{Range}(A) = \{ v \in \mathbb{R}^5 : v = Ax \text{ for some } x \}.$$

(d) Describe, in words, the row space of $A$ (this is the column space of $A^T$):

$$\text{Range}(A^T) = \{ v \in \mathbb{R}^5 : v = A^Tx \text{ for some } x \}.$$
Problem 9: Eigenvalues and eigenvectors. Suppose that two \( n \times n \) matrices \( A \) and \( B \) have the same eigenvectors, \( v_1, \ldots, v_n \), but different sets of eigenvalues. Matrix \( A \) has eigenvalues \( \lambda_1, \ldots, \lambda_n \) with respective eigenvectors \( v_1, \ldots, v_n \). Matrix \( B \) has eigenvalues \( \gamma_1, \ldots, \gamma_n \) with respective eigenvectors \( v_1, \ldots, v_n \).

(a) What are the eigenvalues and eigenvectors of \( A + B \)? *support your answer*

(b) What are the eigenvalues and eigenvectors of \( AB \)? *support your answer*

(c) What are the eigenvalues and eigenvectors of \( A^{-1}B \)? *support your answer*
Problem 10: Orthogonal projections: Let

\[
p_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 4 \\ -2 \\ -6 \\ -7 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 3 \\ 4 \\ -2 \\ 1 \end{bmatrix},
\]

and

\[
x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 7 \end{bmatrix}.
\]

Find a decomposition of \( x \) into \( x = x^* + x_e \) where \( x^* \) is in the span of \( p_1, p_2, p_3 \), i.e., where \( x^* = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 \) for some suitable choice of \( \alpha_1, \alpha_2, \alpha_3 \). Make sure to give both \( x^* \) and \( x_e \), and show your work/describe your method, even if you use a computer to help with the calculations.
Additional workspace:
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