GEORGIA INSTITUTE OF TECHNOLOGY
School of Electrical and Computer Engineering

Pre-test

Due: January 17, 2018 Course: ECE 6254

Name: ______________________________
Last, First

- Open book/internet.
- No time limit.
- The test is worth 100 points. There are ten questions, and each one is worth 10 points. In multi-part questions, each part will be weighted equally.
- All work should be performed on the test itself. If more space is needed, use the backs of the pages.
- This test will be conducted under the rules and guidelines of the Georgia Tech Honor Code and no cheating will be tolerated (i.e., no discussing the test with other students).
- Make sure to look at the question titles, they will sometimes provide valuable hints.
- Please contact Prof. Anderson (in person or via email) directly with any questions if you are unclear what a question is asking.

Problem 1: Random variables. Suppose that two independent random variables X and Y are distributed according to

$$X \sim \text{Uniform}(0, 3) \quad Y \sim \text{Uniform}(1, 4)$$

What is the probability that $X < Y$.

Problem 2: Independence: Suppose that the probability that A occurs is 0.4 and the probability that **both** A **and** B occur is 0.25. If A and B are independent events, what is the probability that **neither** A **nor** B occur?

Problem 3: Conditional probability density functions and derived distributions:
Suppose that X and Y have joint pdf given by

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-2x-y} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) What are the marginal probability density functions for X and Y?

(b) What is the conditional probability density function $f_{X|Y}(x|y)$?

Problem 4: The median and the cumulative distribution function: Let M be the number of miles your electric car can drive without running out of electricity, and suppose that M has probability density function given by

$$f_M(m) = \begin{cases} \frac{e^{-m/301}}{301} & m \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

What is the median range for your car? That is, how far can you drive before there is a 50% chance that your battery runs out?

Problem 5: Bayes rule and normal random variables: Let X be the score of a randomly selected student in this class on an exam. Let Y denote the mean score of the class. If we knew Y, suppose that a reasonable model for X would be that X is normal with mean Y and variance 20, that is,[1]

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi 20}} e^{-(x-y)^2/40}$$

Suppose that I think that the mean for the exam is probably between 70 and 85, *i.e.*,

$$f_Y(y) = \begin{cases} \frac{1}{15} & 70 \le y \le 85 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that I randomly select a quiz, grade it, and it turns out to receive a 100. Using Bayes rule, how should I update my distribution for Y? That is, what is $f_{Y|X}(y|X = 100)$? (Simplify as much as possible, but you may leave your answer in terms of the standard normal cumulative distribution function Φ if you wish.)

[1] Note that this pdf technically allows some probability of scores above 100 and below 0. You should just ignore this for now.

Problem 6: Joint probability density functions: The *correlation coefficient* $\rho(X,Y)$ between a pair of random variables X and Y is given by

$$\rho(X,Y) = \frac{E\Big[\,(X - E[X])\cdot(Y - E[Y])\,\Big]}{\sigma_X \sigma_Y}.$$

Suppose that X and Y have joint pdf given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{4}x^2(1-y) & 0 \le x \le 2, 0 \le y \le 1 \\ 0 & \text{else.} \end{cases}$$

for any x, y. What is ρ in this case?

Problem 7: Linear Algebra: Pythagoras?

(a) Under what conditions on $\boldsymbol{x}$ and $\boldsymbol{y}$ is it true that

$$\|\boldsymbol{x}+\boldsymbol{y}\|_2^2 = \|\boldsymbol{x}\|_2^2 + \|\boldsymbol{y}\|_2^2 \text{ ?}$$

(b) Under what conditions on $\boldsymbol{x}$ and $\boldsymbol{y}$ is it true that

$$\|\boldsymbol{x}+\boldsymbol{y}\|_2 = \|\boldsymbol{x}\|_2 + \|\boldsymbol{y}\|_2 \text{ ?}$$

Problem 8: Singular value decomposition.: Let

$$\boldsymbol{A} = \begin{bmatrix} -2 & 2 & 2 & -2 & 0 \\ 2 & -2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

(a) What is $\operatorname{rank}(\boldsymbol{A})$?

(b) Using Python or MATLAB (or whatever) find the singular value decomposition of $\boldsymbol{A}$. That is, find matrices $\boldsymbol{U}, \boldsymbol{\Sigma}, \boldsymbol{V}$ such that

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{T}}$$

and $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} = \mathbf{I}$, $\boldsymbol{V}^{\mathrm{T}}\boldsymbol{V} = \mathbf{I}$, and Σ has non-negative entries along its diagonal and is zero elsewhere.

(c) Describe, in words, the column space (or range) of $\boldsymbol{A}$:

$$\mathrm{Range}(\boldsymbol{A}) = \{\boldsymbol{v} \in \mathbb{R}^5 \ : \ \boldsymbol{v} = \boldsymbol{A}\boldsymbol{x} \text{ for some } \boldsymbol{x}\}.$$

(d) Describe, in words, the row space of $\boldsymbol{A}$ (this is the column space of $\boldsymbol{A}^{\mathrm{T}}$):

$$\mathrm{Range}(\boldsymbol{A}^{\mathrm{T}}) = \{\boldsymbol{v} \in \mathbb{R}^5 \ : \ \boldsymbol{v} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{x} \text{ for some } \boldsymbol{x}\}.$$

Problem 9: Eigenvalues and eigenvectors. Suppose that two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvectors, $\mathbf{v}_1, \ldots, \mathbf{v}_n$, but different sets of eigenvalues. Matrix $\mathbf{A}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ with respective eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Matrix $\mathbf{B}$ has eigenvalues $\gamma_1, \ldots, \gamma_n$ with respective eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

(a) What are the eigenvalues and eigenvectors of $\mathbf{A} + \mathbf{B}$? *support your answer*

(b) What are the eigenvalues and eigenvectors of $\mathbf{AB}$? *support your answer*

(c) What are the eigenvalues and eigenvectors of $\mathbf{A}^{-1}\mathbf{B}$? *support your answer*

Problem 10: Orthogonal projections: Let

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} 4 \\ -2 \\ -6 \\ -7 \end{bmatrix} \qquad \mathbf{p}_3 = \begin{bmatrix} 3 \\ 4 \\ -2 \\ 1 \end{bmatrix}$$

and

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 7 \end{bmatrix}.$$

Find a decomposition of $\mathbf{x}$ into $\mathbf{x} = \mathbf{x}^* + \mathbf{x}_e$ where $\mathbf{x}^*$ is in the span of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, i.e., where $\mathbf{x}^* = \alpha_1 \mathbf{p}_1 + \alpha_2 \mathbf{p}_2 + \alpha_3 \mathbf{p}_3$ for some suitable choice of $\alpha_1, \alpha_2, \alpha_3$. Make sure to give both $\mathbf{x}^*$ and $\mathbf{x}_e$, and show your work/describe your method, even if you use a computer to help with the calculations.

Additional workspace:

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