Regression recap

Recall that in regression we are given training data

$$(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n)$$

where $\mathbf{x}_i \in \mathbb{R}^d$ and $y_i \in \mathbb{R}$

In linear regression we assume that we are trying to estimate a function of the form

$$f(\mathbf{x}) = \boldsymbol{\beta}^T \mathbf{x} + \beta_0$$

where $\boldsymbol{\beta} \in \mathbb{R}^d$, $\beta_0 \in \mathbb{R}$

Least squares regression: Select $\boldsymbol{\beta}, \beta_0$ to minimize

$$\mathrm{SSE}(\boldsymbol{\beta}, \beta_0) := \sum_{i=1}^{n} \left(y_i - \boldsymbol{\beta}^T \mathbf{x}_i - \beta_0\right)^2$$

Least squares

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \boldsymbol{A} = \begin{bmatrix} 1 & x_1(1) & \cdots & x_1(d) \\ 1 & x_2(1) & \cdots & x_2(d) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n(1) & \cdots & x_n(d) \end{bmatrix} \quad \boldsymbol{\theta} = \begin{bmatrix} \beta_0 \\ \beta(1) \\ \vdots \\ \beta(d) \end{bmatrix}$$

$$\mathrm{SSE}(\boldsymbol{\theta}) = \sum_{i=1}^{n} (y_i - \boldsymbol{\beta}^T \mathbf{x}_i - \beta_0)^2 = \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{\theta}\|_2^2$$

Minimizer given by

$$\widehat{\boldsymbol{\theta}} = \left(\boldsymbol{A}^T \boldsymbol{A}\right)^{-1} \boldsymbol{A}^T \boldsymbol{y}$$

provided that $\boldsymbol{A}^T \boldsymbol{A}$ is ***nonsingular***

Regularization and regression

Overfitting occurs as the number of features d begins to approach the number of observations n

In this regime, we have *too many degrees of freedom*

Idea: penalize candidate solutions for using too many features

One candidate regularizer: $r(\boldsymbol{\theta}) = \|\boldsymbol{\theta}\|_2^2$

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{\theta}\|_2^2 + \lambda \|\boldsymbol{\theta}\|_2^2$$

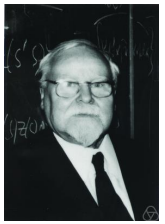
$\lambda > 0$ is a "tuning parameter" that controls the tradeoff between fit and complexity

Tikhonov regularization

This is one example of a more general technique called ***Tikhonov regularization***

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{\theta}\|_2^2 + \|\boldsymbol{\Gamma}\boldsymbol{\theta}\|_2^2$$

(Note that λ has been replaced by the matrix $\boldsymbol{\Gamma}$)



Solution:

$$\widehat{\boldsymbol{\theta}} = \left(\boldsymbol{A}^T \boldsymbol{A} + \boldsymbol{\Gamma}^T \boldsymbol{\Gamma}\right)^{-1} \boldsymbol{A}^T \boldsymbol{y}$$

$$\boldsymbol{\Gamma} = \sqrt{\lambda}\boldsymbol{I} \qquad \widehat{\boldsymbol{\theta}} = \left(\boldsymbol{A}^T \boldsymbol{A} + \lambda \boldsymbol{I}\right)^{-1} \boldsymbol{A}^T \boldsymbol{y}$$

for suitable choice of λ, always well-conditioned

Ridge regression

In the context of regression, Tikhonov regularization has a special name: ***ridge regression***

Ridge regression is essentially exactly what we have been talking about, but in the special case where

$$\boldsymbol{\Gamma} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{\lambda} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{\lambda} \end{bmatrix}$$

We are penalizing all coefficients in $\boldsymbol{\beta}$ equally, but not penalizing the offset β_0

Alternative regularizers

- Akaike information criterion (AIC)
- Bayesian information criterion (BIC)

$$r(\boldsymbol{\theta}) \approx \|\boldsymbol{\theta}\|_0 := |\text{supp}(\boldsymbol{\theta})|$$

- Least absolute shrinkage and selection operator (LASSO)

$$r(\boldsymbol{\theta}) = \|\boldsymbol{\theta}\|_1 = \sum_j |\theta(j)|$$

  - also results in shrinkage, but where all coordinates are shrunk by the same amount (in case of an orthobasis)
  - promotes sparsity
  - can think of $\|\boldsymbol{\theta}\|_1$ as a more computationally tractable replacement for $\|\boldsymbol{\theta}\|_0$

The LASSO

LASSO

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{\theta}\|_2^2 + \lambda\|\boldsymbol{\theta}\|_1$$

Can also be stated in a constrained form

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{\theta}\|_2^2 \quad \text{s.t.} \quad \|\boldsymbol{\theta}\|_1 \leq \tau$$

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \|\boldsymbol{\theta}\|_1 \quad \text{s.t.} \quad \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{\theta}\|_2^2 \leq \sigma$$

For Tikhonov, we have a closed form solution, but LASSO requires solving an optimization problem

Note: Just like in ridge regression, in practice we may just want to penalize the elements of $\boldsymbol{\beta}$ (not β_0)

Sparsity and the LASSO

One can show (see supplemental notes) that if we have a data set of size n, then the solution to the LASSO $\widehat{\boldsymbol{\theta}}$ will have at most n nonzeros (for any possible dataset / $\boldsymbol{A}$)

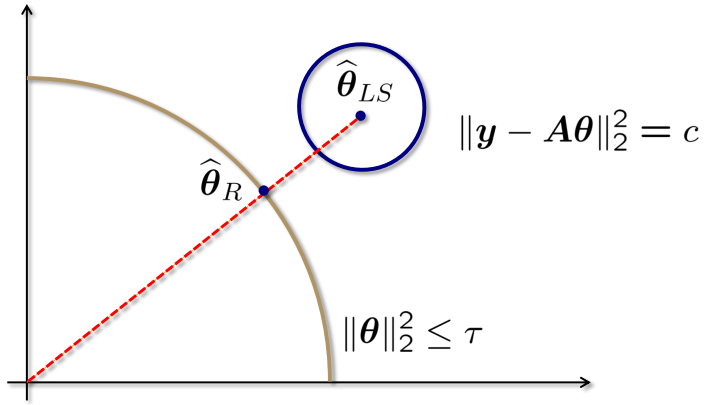
This is a nice property when $n \ll d$, since in this setting we are ***very*** susceptible to overfitting

- fewer observations than unknowns
- $\boldsymbol{A}$ has nontrivial nullspace
- we can achieve $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{\theta}$, with infinitely many different choices of $\boldsymbol{\theta}$ and no obvious way to know which one is best
- limiting the number of nonzeros addresses this problem

In practice, the number of nonzeros is usually much smaller than n

Tikhonov versus least squares

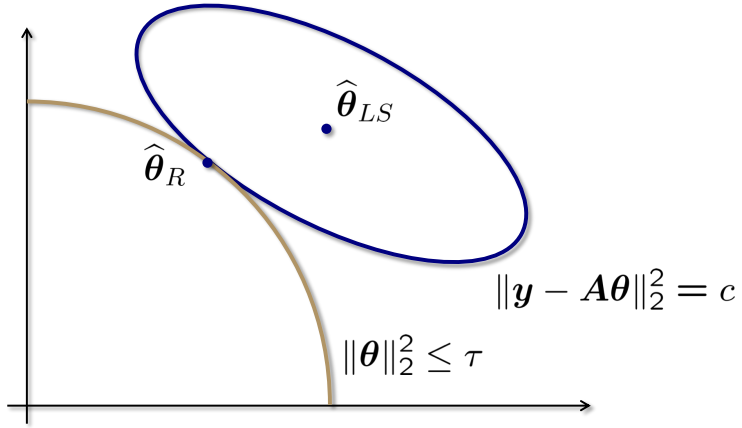
Assume $\Gamma = I$ and that A has orthonormal columns



Tikhonov regularization is equivalent to shrinking the least squares solution towards the origin

Tikhonov versus least squares

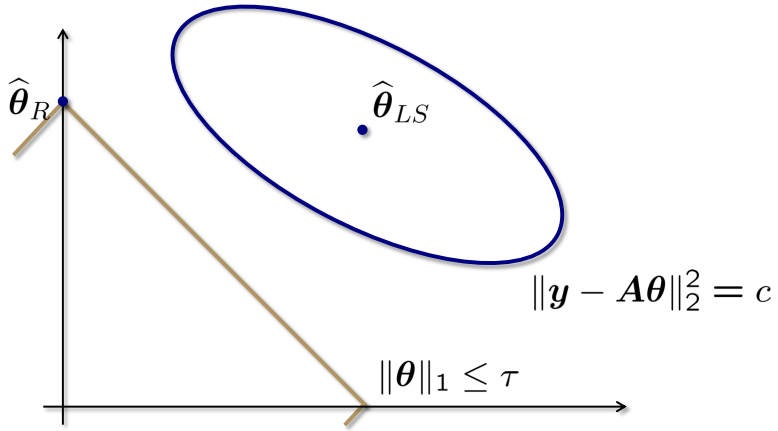
In general, we have this picture



Tikhonov regularization still shrinking the least squares solution towards the origin

Lasso versus least squares

For the LASSO we get something like this…



LASSO still shrinking the least squares solution towards the origin, but now in a way that promotes sparsity

A general approach to regression

Least squares, ridge regression, and the LASSO call all be viewed as particular instances of the following general approach to regression

$$\widehat{\theta} = \arg\min_{\theta} \; L(\theta) + \lambda r(\theta)$$

- $L(\theta)$, often called the ***loss function***, enforces data fidelity

$$f_\theta(\mathbf{x}_i) \approx y_i$$

- $r(\theta)$ is a ***regularizer*** which serves to quantify the "complexity" of θ

We have seen some examples of regularizers, what about other loss functions?

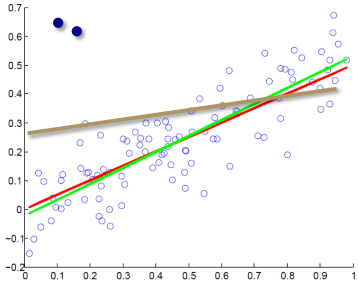
Outliers in regression

The squared error loss function is sensitive to ***outliers***

If $f(\mathbf{x}_i) - y_i$ is small, then $(f(\mathbf{x}_i) - y_i)^2$ is not too large

But if $f(\mathbf{x}_i) - y_i$ is big, then $(f(\mathbf{x}_i) - y_i)^2$ is ***really*** big

Normally this is not a bad property - we want to penalize big errors - but this can make us very sensitive to large outliers



Robust regression

What else could we do aside from least squares?

Mean absolute error

$$L_{AE}(r) = |r|$$

Huber loss

$$L_H(r) = \begin{cases} \frac{1}{2}r^2 & \text{if } |r| \le c \\ c|r| - \frac{c^2}{2} & \text{if } |r| > c \end{cases}$$

ϵ-insensitive loss

$$L_\epsilon(r) = \begin{cases} 0 & \text{if } |r| \le c \\ |r| - \epsilon & \text{if } |r| > \epsilon \end{cases}$$

Regularized robust regression

Suppose we combine this loss with an ℓ_2 regularizer

$$\widehat{\boldsymbol{\beta}}, \beta_0 = \underset{(\boldsymbol{\beta},\beta_0)}{\arg\min} \sum_{i=1}^{n} L_\epsilon(y_i - (\boldsymbol{\beta}^T \mathbf{x}_i + \beta_0)) + \frac{\lambda}{2}\|\boldsymbol{\beta}\|_2^2$$

Note that the ϵ-insensitive loss has no penalty as long as your prediction is within a "margin" of ϵ

This looks like an SVM...

Support vector regression

The previous problem can also be cast in the following dual form

$$\min_{\boldsymbol{\alpha},\boldsymbol{\alpha}^*} \sum_i ((\epsilon - y_i)\alpha_i^* + (\epsilon + y_i)\alpha_i) + \frac{1}{2}\sum_{i,j}(\alpha_i^* - \alpha_i)(\alpha_j^* - \alpha_j)\mathbf{x}_j^T\mathbf{x}_i$$

$$\text{subject to} \quad 0 \le \alpha_i^*, \alpha_i \le \tfrac{1}{\lambda}$$

$$\sum_i (\alpha_i^* - \alpha_i) = 0$$

$$\alpha_i^* \alpha_i = 0$$

The solution has the form $\widehat{f}(\mathbf{x}) = \sum_i (\widehat{\alpha}_i^* - \widehat{\alpha}_i)\mathbf{x}_i^T\mathbf{x} + \beta_0$

Can we kernelize regression?

In order to "kernelize" an algorithm, the general approach consists of three main steps:

1. Show that the training process only involves the training data via inner products (i.e., $\mathbf{x}_i^T \mathbf{x}_j$)
2. Show that applying the decision rule to a new $\mathbf{x}$ only involves computing inner products (i.e., $\mathbf{w}^T \mathbf{x}$)
3. Replace all inner products with evaluations of the kernel function $k(\cdot, \cdot)$

This approach extends well beyond SVMs

Kernel support vector regression

$$\min_{\boldsymbol{\alpha}, \boldsymbol{\alpha}^*} \sum_i ((\epsilon - y_i)\alpha_i^* + (\epsilon + y_i)\alpha_i) + \frac{1}{2}\sum_{i,j}(\alpha_i^* - \alpha_i)(\alpha_j^* - \alpha_j)\mathbf{x}_j^T \mathbf{x}_i$$

$$\text{subject to} \quad 0 \leq \alpha_i^*, \alpha_i \leq \tfrac{1}{\lambda}$$

$$\sum_i (\alpha_i^* - \alpha_i) = 0$$

$$\alpha_i^* \alpha_i = 0$$

$$\widehat{f}(\mathbf{x}) = \sum_i (\widehat{\alpha}_i^* - \widehat{\alpha}_i)\mathbf{x}_i^T \mathbf{x} + \beta_0$$

Straightforward to kernelize

Kernelized LASSO?

Can we kernelize this?

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{\theta}\|_2^2 + \lambda\|\boldsymbol{\theta}\|_1$$

Not exactly…

Nevertheless, we can assert (with no justification) that

$$\widehat{\boldsymbol{\theta}} = \sum_i \alpha_i \mathbf{x}_i$$

and then replace $\|\boldsymbol{\theta}\|_1$ with $\|\boldsymbol{\alpha}\|_1$

This yields an algorithm that can be easily kernelized, although it is really something different than the LASSO

- promotes sparsity in $\boldsymbol{\alpha}$, not $\boldsymbol{\theta}$

Ridge regression revisited

Ridge regression: Given $\{(\mathbf{x}_i, y_i)\}_{i=1}^n$, $\mathbf{x}_i \in \mathbb{R}^d, y_i \in \mathbb{R}$

$$(\widehat{\boldsymbol{\beta}}, \widehat{\beta}_0) = \arg\min_{\boldsymbol{\beta}, \beta_0} \sum_{i=1}^n (y_i - \boldsymbol{\beta}^T \mathbf{x}_i - \beta_0)^2 + \lambda\|\boldsymbol{\beta}\|_2^2$$

Solution: $\dfrac{\partial}{\partial \beta_0} = -2\sum_{i=1}^n (y_i - \boldsymbol{\beta}^T \mathbf{x}_i - \beta_0) = 0$

$$\widehat{\beta}_0 = \frac{1}{n}\sum_{i=1}^n y_i - \widehat{\boldsymbol{\beta}}^T \mathbf{x}_i$$

$$= \bar{y} - \widehat{\boldsymbol{\beta}}^T \bar{\mathbf{x}}$$

$$\bar{y} = \frac{1}{n}\sum_i y_i$$

$$\bar{\mathbf{x}} = \frac{1}{n}\sum_i \mathbf{x}_i$$

Ridge regression revisited

Plugging this back in we are left to minimize

$$\sum_{i=1}^{n}(y_i - \bar{y} - \boldsymbol{\beta}^T(\mathbf{x}_i - \bar{\mathbf{x}}))^2 + \lambda\|\boldsymbol{\beta}\|_2^2$$

with respect to $\boldsymbol{\beta}$

$$\widehat{\boldsymbol{\beta}} = (\mathbf{A}^T\mathbf{A} + \lambda\mathbf{I})^{-1}\mathbf{A}^T\widetilde{\mathbf{y}}$$

$$\mathbf{A} = \begin{bmatrix}(\mathbf{x}_1 - \bar{\mathbf{x}})^T \\ \vdots \\ (\mathbf{x}_n - \bar{\mathbf{x}})^T\end{bmatrix} \qquad \widetilde{\mathbf{y}} = \begin{bmatrix}y_1 - \bar{y} \\ \vdots \\ y_n - \bar{y}\end{bmatrix}$$

Kernel ridge regression

$$\widehat{\boldsymbol{\beta}} = (\mathbf{A}^T\mathbf{A} + \lambda\mathbf{I})^{-1}\mathbf{A}^T\widetilde{\mathbf{y}}$$

$$\mathbf{A} = \begin{bmatrix}(\mathbf{x}_1 - \bar{\mathbf{x}})^T \\ \vdots \\ (\mathbf{x}_n - \bar{\mathbf{x}})^T\end{bmatrix} \qquad \widetilde{\mathbf{y}} = \begin{bmatrix}y_1 - \bar{y} \\ \vdots \\ y_n - \bar{y}\end{bmatrix}$$

$$\widehat{f}(\mathbf{x}) = \bar{y} + \widehat{\boldsymbol{\beta}}^T(\mathbf{x} - \bar{\mathbf{x}})$$

Can we express ridge regression in terms of inner products $\langle \mathbf{x}_i, \mathbf{x}_j \rangle$ and $\langle \mathbf{x}_i, \mathbf{x} \rangle$?

Not immediately. $[\boldsymbol{A}^T\boldsymbol{A}](i,j) \neq \mathbf{x}_i^T\mathbf{x}_j$

Woodbury matrix inversion identity

$$(\mathbf{P} + \mathbf{QRS})^{-1} = \mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{Q}(\mathbf{R}^{-1} + \mathbf{SP}^{-1}\mathbf{Q})^{-1}\mathbf{SP}^{-1}$$

$$\mathbf{P} = \lambda\mathbf{I} \quad \mathbf{Q} = \mathbf{A}^T \quad \mathbf{R} = \mathbf{I} \quad \mathbf{S} = \mathbf{A}$$

$$\begin{aligned}(\lambda\mathbf{I} + \mathbf{A}^T\mathbf{A})^{-1} &= \frac{1}{\lambda}\mathbf{I} - \frac{1}{\lambda}\mathbf{I}\mathbf{A}^T\left(\mathbf{I} + \frac{1}{\lambda}\mathbf{A}\mathbf{A}^T\right)^{-1}\mathbf{A}\frac{1}{\lambda} \\ &= \frac{1}{\lambda}\left[\mathbf{I} - \mathbf{A}^T(\lambda\mathbf{I} + \mathbf{A}\mathbf{A}^T)^{-1}\mathbf{A}\right]\end{aligned}$$

$$(\lambda\mathbf{I} + \mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\widetilde{\mathbf{y}} = \frac{1}{\lambda}\left[\mathbf{A}^T - \mathbf{A}^T(\lambda\mathbf{I} + \mathbf{A}\mathbf{A}^T)^{-1}\mathbf{A}\mathbf{A}^T\right]\widetilde{\mathbf{y}}$$

Kernelizing ridge regression

$$(\lambda\mathbf{I} + \mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\widetilde{\mathbf{y}} = \frac{1}{\lambda}\left[\mathbf{A}^T - \mathbf{A}^T(\lambda\mathbf{I} + \mathbf{A}\mathbf{A}^T)^{-1}\mathbf{A}\mathbf{A}^T\right]\widetilde{\mathbf{y}}$$

$$\begin{aligned}K(i,j) &= (\mathbf{x}_i - \bar{\mathbf{x}})^T(\mathbf{x}_j - \bar{\mathbf{x}}) \\ &= \mathbf{x}_i^T\mathbf{x}_j - \frac{1}{n}\sum_{r=1}^{n}\mathbf{x}_i^T\mathbf{x}_r - \frac{1}{n}\sum_{s=1}^{n}\mathbf{x}_s^T\mathbf{x}_j + \frac{1}{n^2}\sum_{r,s=1}^{n}\mathbf{x}_r^T\mathbf{x}_s\end{aligned}$$

$$(\lambda\mathbf{I} + \mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\widetilde{\mathbf{y}} = \frac{1}{\lambda}\left[\mathbf{A}^T - \mathbf{A}^T(\lambda\mathbf{I} + \mathbf{K})^{-1}\mathbf{K}\right]\widetilde{\mathbf{y}}$$

Kernelizing ridge regression

What about the remaining $\mathbf{A}^T$?

$$\begin{aligned}\widehat{f}(\mathbf{x}) &= \bar{y} + \widehat{\boldsymbol{\beta}}^T(\mathbf{x} - \bar{\mathbf{x}}) \\ &= \bar{\mathbf{y}} + \frac{1}{\lambda}\tilde{\mathbf{y}}^T\left[\mathbf{A} - \mathbf{K}(\lambda\mathbf{I} + \mathbf{K})^{-1}\mathbf{A}\right](\mathbf{x} - \bar{\mathbf{x}}) \\ &= \bar{\mathbf{y}} + \frac{1}{\lambda}\tilde{\mathbf{y}}^T\left[\mathbf{I} - \mathbf{K}(\lambda\mathbf{I} + \mathbf{K})^{-1}\mathbf{I}\right]\mathbf{k}(\mathbf{x})\end{aligned}$$

where $\mathbf{k}(\mathbf{x}) = \begin{bmatrix}(\mathbf{x}_1 - \bar{\mathbf{x}})^T(\mathbf{x} - \bar{\mathbf{x}}) \\ \vdots \\ (\mathbf{x}_n - \bar{\mathbf{x}})^T(\mathbf{x} - \bar{\mathbf{x}})\end{bmatrix}$

$$[k(\mathbf{x})](i) = \mathbf{x}_i^T\mathbf{x} - \frac{1}{n}\sum_{j=1}^{n}\mathbf{x}_i^T\mathbf{x}_j - \frac{1}{n}\sum_{j=1}^{n}\mathbf{x}^T\mathbf{x}_j + \frac{1}{n^2}\sum_{j,k=1}^{n}\mathbf{x}_j^T\mathbf{x}_k$$

Homogenous kernel ridge regression

For many kernels, $\Phi(\mathbf{x})$ already contains a constant component, in which case we often omit β_0

- inhomogenous polynomial kernel
- Gaussian kernel does not seem to require a constant

In this case, the kernel ridge regression solution becomes

$$\widehat{f}(\mathbf{x}) = \mathbf{y}^T(\mathbf{K} + \lambda\mathbf{I})^{-1}\mathbf{k}(\mathbf{x})$$

where $\mathbf{y} = \left[y_1, \cdots, y_n\right]^T$

$$\mathbf{k}(\mathbf{x}) = \left[k(\mathbf{x}_1, \mathbf{x}), \cdots, k(\mathbf{x}_n, \mathbf{x})\right]^T$$

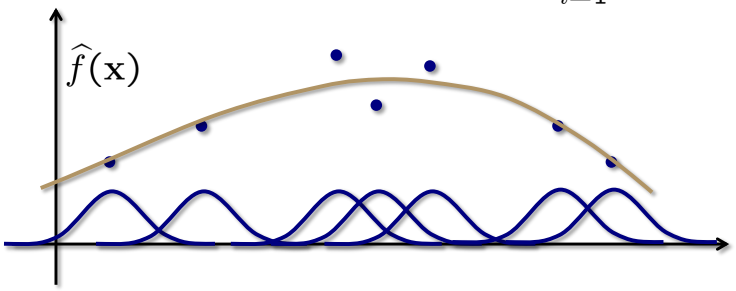
$$\mathbf{K} = \begin{bmatrix}k(\mathbf{x}_1, \mathbf{x}_1) & \cdots & k(\mathbf{x}_1, \mathbf{x}_n) \\ \vdots & \ddots & \vdots \\ k(\mathbf{x}_n, \mathbf{x}_1) & \cdots & k(\mathbf{x}_n, \mathbf{x}_n)\end{bmatrix}$$

Example: Gaussian kernel

$$k(\mathbf{u}, \mathbf{v}) = \exp\left(-\frac{\|\mathbf{u} - \mathbf{v}\|_2^2}{2\sigma^2}\right)$$

If we omit β_0, then $\widehat{f}(\mathbf{x}) = \mathbf{y}^T(\mathbf{K} + \lambda\mathbf{I})^{-1}\mathbf{k}(\mathbf{x})$

$$= \boldsymbol{\alpha}^T\mathbf{k}(\mathbf{x}) = \sum_{i=1}^{n}\alpha(i)k(\mathbf{x}, \mathbf{x}_i)$$



Loss functions and regularization

We have talked about a whole range of algorithms for regression that can be viewed through the lens of minimizing a loss function plus a regularization term

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \; L(\boldsymbol{\theta}) + \lambda r(\boldsymbol{\theta})$$

Does this viewpoint also apply to classification?

Regularized logistic regression

Everything we have said so far about least squares regression can be extended to many classification problems

For example, in logistic regression we can replace

$$\min_{\boldsymbol{\theta}} \; -\ell(\boldsymbol{\theta})$$

with

$$\min_{\boldsymbol{\theta}} \; -\ell(\boldsymbol{\theta}) + \lambda\|\boldsymbol{\theta}\|_2^2$$

Has a similar interpretation to least squares regularization

- makes the Hessian matrix well conditioned
- super useful when the number of observations is small
- also helpful when data is separable

Regularization for linear classification

The kinds of regularization we have talked about can also give us a new way to think about designing linear classifiers

Goal (ideal): Find $(\mathbf{w}, b)$ minimizing

$$\frac{1}{n}\sum_{i=1}^{n} 1_{\{y_i(\mathbf{w}^T\mathbf{x}_i+b)<0\}}$$

This is actually much harder than it sounds and is not computationally tractable for large problems

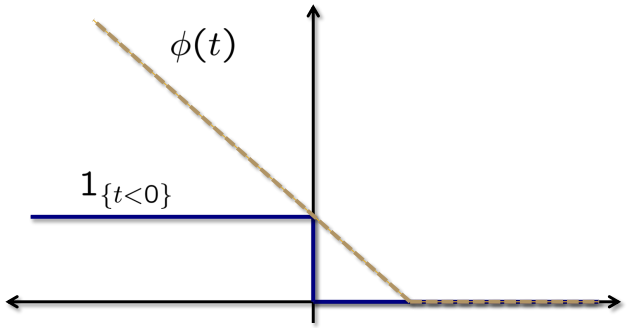
Instead, we can consider replacing this with

$$\frac{1}{n}\sum_{i=1}^{n} \phi(y_i(\mathbf{w}^T\mathbf{x}_i + b))$$

where $\phi(t)$ is some upper bound on $1_{\{t<0\}}$

Hinge loss

Let's take $\phi(t) = \max\{0, 1-t\} =: (1-t)_+$



Adding regularization

Let's try to minimize

$$\frac{1}{n}\sum_{i=1}^{n}(1 - y_i(\mathbf{w}^T\mathbf{x}_i + b))_+$$

but to prevent overfitting, let's add a regularization penalty on $\mathbf{w}$

$$\min_{\mathbf{w},b} \; \frac{1}{n}\sum_{i=1}^{n}(1 - y_i(\mathbf{w}^T\mathbf{x}_i + b))_+ + \frac{\lambda}{2}\|\mathbf{w}\|_2^2$$

Soft-margin hyperplane

Compare this to the optimization problem we considered previously for the optimal soft-margin hyperplane:

$$\min_{\mathbf{w},b} \ \frac{1}{n}\sum_{i=1}^{n}(1 - y_i(\mathbf{w}^T\mathbf{x}_i + b))_+ + \frac{\lambda}{2}\|\mathbf{w}\|_2^2$$

vs

$$\begin{aligned}\min_{\mathbf{w},b,\boldsymbol{\xi}} \ & \frac{1}{2}\|\mathbf{w}\|_2^2 + \frac{C}{n}\sum_{i=1}^{n}\xi_i \\ \text{s.t.} \ & y_i(\mathbf{w}^T\mathbf{x}_i + b) \geq 1 - \xi_i \quad i = 1, \ldots, n \\ & \xi_i \geq 0 \quad i = 1, \ldots, n\end{aligned}$$

As you all just showed, these are equivalent!